Lecture 6: Shallow water dynamics and gravity waves

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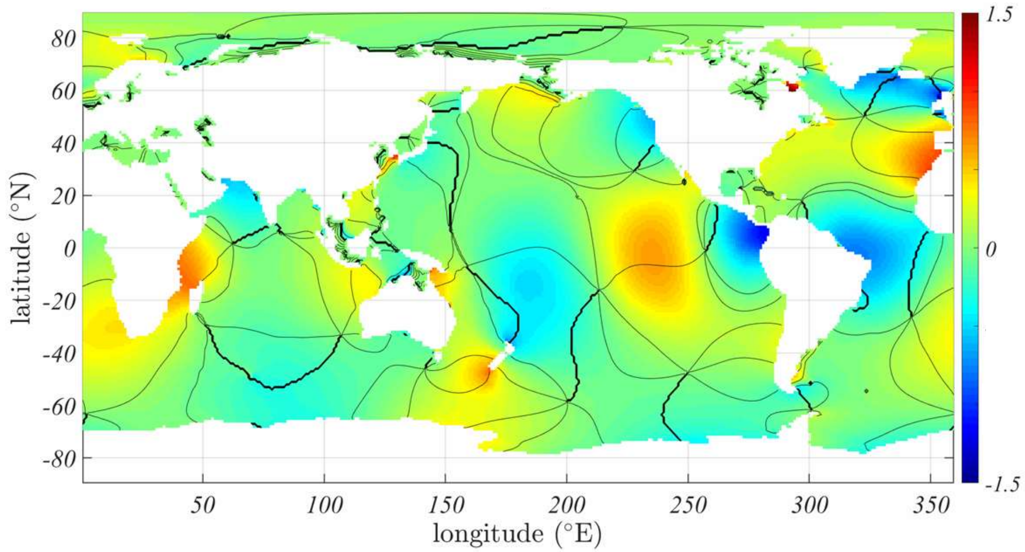


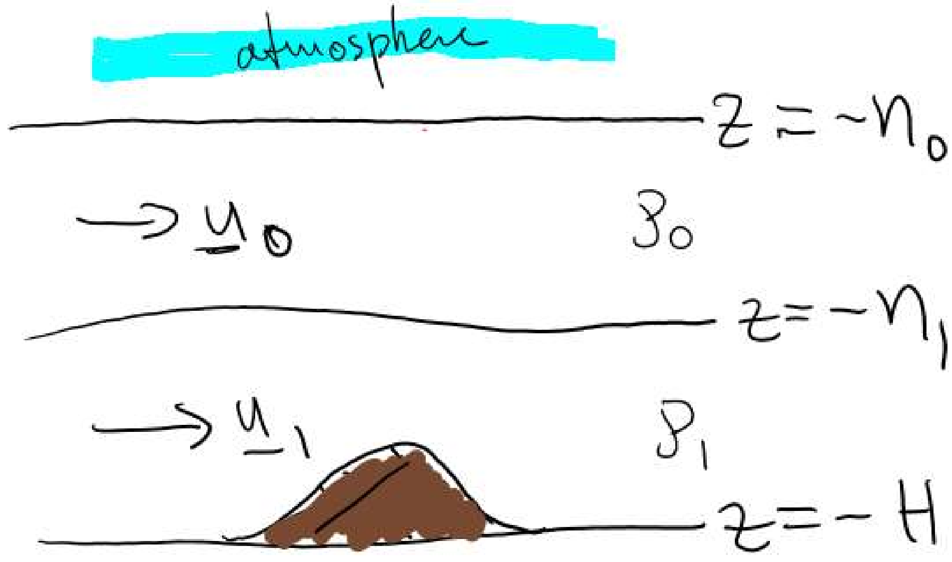
Derivation of 2-layer shallow water equations

Hydrostatic momentum equations

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial z} = \rho g$$



$$p_0 = \rho_0 g(-\eta_0 - z)$$

$$p_1 = \rho_0 g(-\eta_0 + \eta_1) + \rho_1 g(-\eta_1 - z)$$

Derivation of 2-layer shallow water equations

$$\frac{\partial p}{\partial z} = \rho g$$

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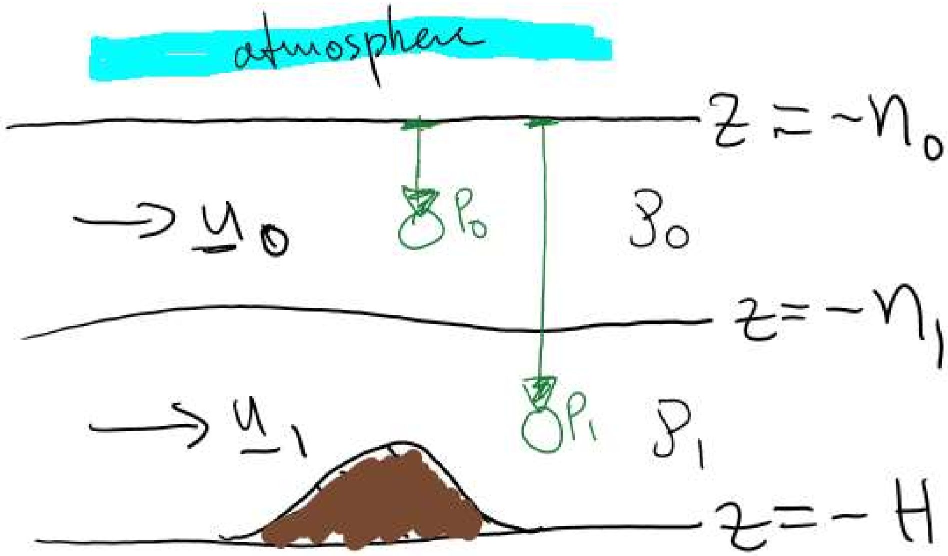


$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Layer 0

$$\frac{Du_0}{Dt} - fv_0 = g\frac{\partial \eta_0}{\partial x}$$

Layer 1

$$\frac{Du_1}{Dt} - fv_1 = g\frac{\partial(\eta_0 - \eta_1)}{\partial x} + g\frac{\rho_1}{\rho_0}\frac{\partial \eta_1}{\partial x}$$

$$= g\frac{\partial \eta_0}{\partial x} + \underbrace{\boxed{g\frac{\rho_1 - \rho_0}{\rho_0}}}_{g'}\frac{\partial \eta_1}{\partial x}$$

Derivation of 2-layer shallow water equations

Conservation of mass

$$h_0 = \eta_1 - \eta_0$$

$$\frac{\partial h_0}{\partial t} = -\nabla\cdot(\text{volume flux})$$
$$= -\nabla\cdot(h_0 \underline{u}_0)$$
$$= -h_0 \nabla\cdot\underline{u}_0 - \underline{u}_0\cdot\nabla h$$

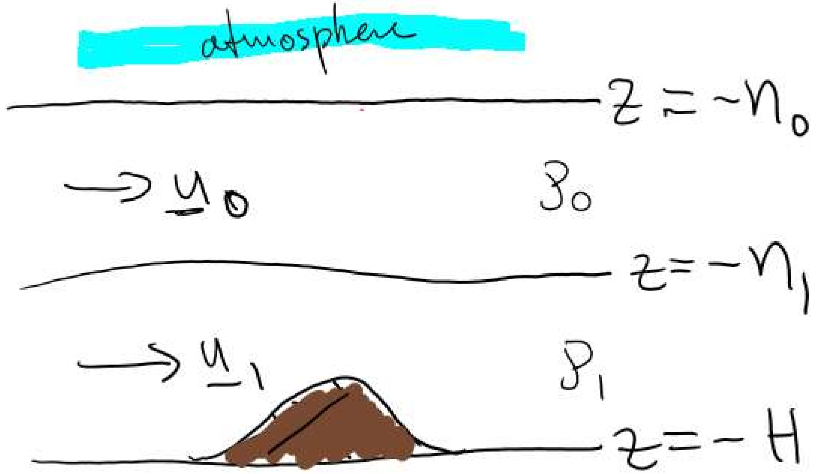
$$\frac{Dh_0}{Dt} = -h_0 \nabla\cdot\mathbf{u_0}$$

Layer 1 similar but $h_1 = H - \eta_1$

2-layer case

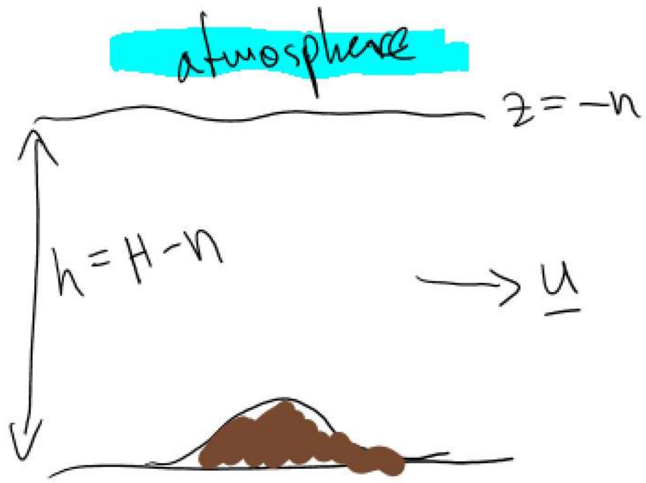
$$\frac{Du_0}{Dt} - fv_0 = g\frac{\partial \eta_0}{\partial x}$$
$$\frac{Dv_0}{Dt} + fu_0 = g\frac{\partial \eta_0}{\partial y}$$
$$\frac{Dh_0}{Dt} = -h_0 \nabla \cdot \mathbf{u_0}$$
$$h_0 = \eta_1 - \eta_0$$

$$\frac{Du_1}{Dt} - fv_1 = g\frac{\partial \eta_0}{\partial x} + g'\frac{\partial \eta_1}{\partial x}$$
$$\frac{Dv_1}{Dt} + fu_1 = g\frac{\partial \eta_0}{\partial y} + g'\frac{\partial \eta_1}{\partial y}$$
$$\frac{Dh_1}{Dt} = -h_1 \nabla \cdot \mathbf{u_1}$$
$$h_1 = H - \eta_1$$



1-layer case

$$\frac{Du}{Dt} - fv = g\frac{\partial \eta}{\partial x}$$
$$\frac{Dv}{Dt} + fu = g\frac{\partial \eta}{\partial y}$$
$$\frac{Dh}{Dt} = -h \nabla \cdot \mathbf{u}$$
$$h = H - \eta$$



Linearisation and wave solutions

Non-linear equations

$$\frac{Du}{Dt} - fv = g\frac{\partial \eta}{\partial x}$$

$$\frac{Dv}{Dt} + fu = g\frac{\partial \eta}{\partial y}$$

$$\frac{Dh}{Dt} = -h\nabla \cdot \mathbf{u}$$

$$h = H - \eta$$

Linearised equations

$$\frac{\partial u}{\partial t} - fv = g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = g\frac{\partial \eta}{\partial y}$$

$$-\frac{\partial \eta}{\partial t} + \nabla \cdot (\underline{u} H) = 0$$

Assumptions

$$\left(\underline{u} \cdot \nabla << \frac{\partial}{\partial t} \Rightarrow \frac{U}{\omega L} << 1\right)$$

$$(\eta << H)$$

Linearisation

Non-linear equations

$$\frac{Du}{Dt} - fv = g\frac{\partial \eta}{\partial x}$$

$$\frac{Dv}{Dt} + fu = g\frac{\partial \eta}{\partial y}$$

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Linearised equations

$$\frac{\partial u}{\partial t} - fv = g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = g\frac{\partial \eta}{\partial y}$$

$$-\frac{\partial \eta}{\partial t} + \nabla \cdot (\underline{u} H) = 0$$

Assumptions

$$\left(\underline{u} \cdot \nabla \ll \frac{\partial}{\partial t} \Rightarrow \frac{U}{\omega L} \ll 1 \right)$$

$$(\eta \ll H)$$

Let $\underline{U} = \underline{u} H$

$$\frac{\partial U}{\partial t} - fV = gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial V}{\partial t} + fU = gH\frac{\partial \eta}{\partial y}$$

$$-\frac{\partial \eta}{\partial t} + \nabla \cdot \underline{U} = 0$$

Depth integrated linearised equations

Wave solutions

$$\frac{\partial u}{\partial t} - fv = gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = gH\frac{\partial \eta}{\partial y}$$

$$-\frac{\partial \eta}{\partial t} + \nabla \cdot \underline{u} = 0$$

Let $\phi = \hat{\phi}\, e^{i(kx+ly-\omega t)}$

$$\Rightarrow \quad -i\omega\hat{u} - f\hat{v} = ikgH\hat{\eta}$$

$$-i\omega\hat{v} + f\hat{u} = ilgH\hat{\eta}$$

$$+i\omega\hat{\eta} + ik\hat{u} + il\hat{v} = 0$$

Dispersion Relation

$$\omega^2 = f^2 + (k^2 + l^2)gH$$

$$\omega = \pm\sqrt{f^2 + K^2 gH}$$

Barotropic gravity waves (BGWs)

How do we generate BGWs?

- Full depth (barotropic) forcing = body force
- E.g. horizontal variations in gravity due to the moon = tidal forces!
- Usually represented in terms of an equilibrium tide = height perturbation that would exist if no flow: $\bar{\eta} = A\cos(\omega_t t + 2\phi)$
- Flow is associated with departures from $\bar{\eta}$
- Let $\eta = \bar{\eta} + \eta'$

$$\frac{\partial u}{\partial t} - fv = gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = gH\frac{\partial \eta}{\partial y}$$

$$-\frac{\partial \eta}{\partial t} + \nabla\cdot\underline{u} = 0$$

$$\frac{\partial u}{\partial t} - fv = gH\frac{\partial \eta'}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = gH\frac{\partial \eta'}{\partial y}$$

$$-\frac{\partial \eta'}{\partial t} + \nabla\cdot\underline{u} = \boxed{+\frac{\partial \bar{\eta}}{\partial t}}$$

Tidal forcing

These are the Tidal Equations
or if on a sphere "Laplace Tidal Equations"

Properties of the tides

- Forced at a particular frequency = 12.4 hour period

$$\omega = \pm\sqrt{f^2 + K^2 gH}$$

- $f \sim \frac{10^{-4}}{s}, \;\; \omega \sim 1.4 \times \frac{10^{-4}}{s}, g \sim 10\,\frac{m}{s^2}, H \sim 4000\; m$
- What is the value of K?
- $\frac{1}{K} \sim$ 2000 km

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How is the size of my ocean basin

related to the amplitude of the tide????

Properties of the tides

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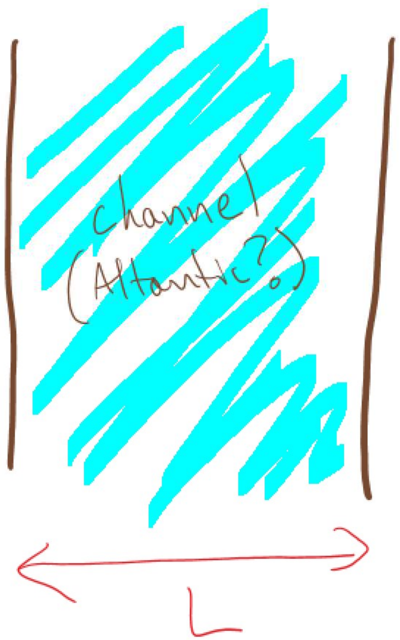

- Consider a simplified channel
- Infinitely long in y
- Width L in x

Properties of the tides

How is the size of my ocean basin related to the amplitude of the tide????

$$\frac{\partial u}{\partial t} - fv = -gH\frac{\partial \eta'}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = 0$$

$$\frac{\partial \eta'}{\partial t} + \frac{\partial u}{\partial x} = -\frac{\partial \bar{\eta}}{\partial t} \quad \Big\{ \bar{\eta} = A\cos\left(\omega t + \frac{2x}{C}\right)$$

$$\boxed{C = 2\pi R_E \cos\theta}$$ Circumference of Earth

$$\bar{\eta} \simeq A\cos(\omega t) - \frac{4Ax}{C}\sin(\omega t) \qquad \left(\frac{2x}{C} \ll \omega t\right)$$

Properties of the tides

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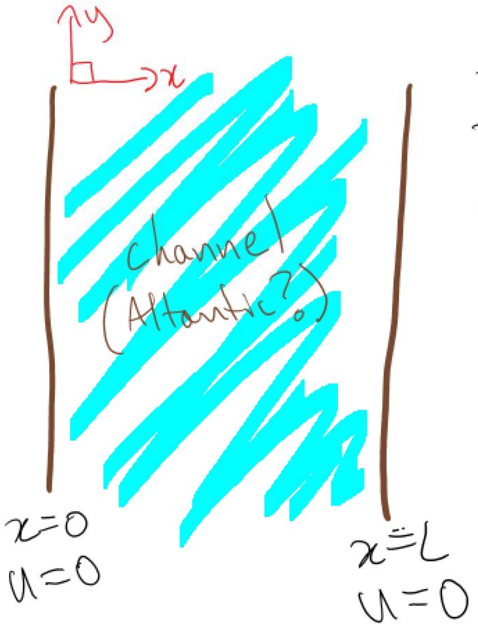

$$\frac{\partial u}{\partial t} - fV = -gH\frac{\partial \eta'}{\partial x}$$

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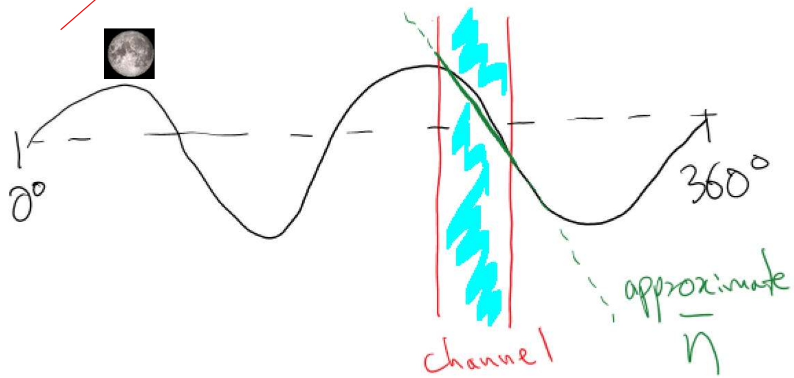

Properties of the tides

How is the size of my ocean basin related to the amplitude of the tide????

Let: $\eta' = \mathrm{Im}(\hat{\eta}' e^{i\omega t})$

$$i\omega \hat{u} - f\hat{v} = -gH \frac{\partial \hat{\eta}}{\partial x}$$

$$i\omega \hat{v} + f\hat{u} = 0 \Rightarrow \hat{v} = -\frac{f}{i\omega}\hat{u}$$

$$i\omega(\hat{\eta}' - \hat{\eta}) + \frac{\partial \hat{u}}{\partial x} = 0$$

Thus: $i\omega \hat{u} + \frac{f^2}{i\omega}\hat{u} = -gH \frac{\partial \hat{\eta}'}{\partial x}$

$$\Rightarrow \hat{u} = -\frac{i\omega}{f^2 - \omega^2} gH \frac{\partial \hat{\eta}'}{\partial x}$$

$$\Rightarrow i\omega(\hat{\eta}' - A) - \frac{i\omega gH}{f^2 - \omega^2}\frac{\partial^2 \hat{\eta}'}{\partial x^2} = 0$$

$$\boxed{\hat{\eta}' - \frac{gH}{f^2 - \omega^2}\frac{\partial^2 \hat{\eta}'}{\partial x^2} = \frac{-4Ax}{C}}$$

where $\hat{u} \sim \frac{\partial \hat{\eta}'}{\partial x} = 0$ at $x = 0, L$

Properties of the tides

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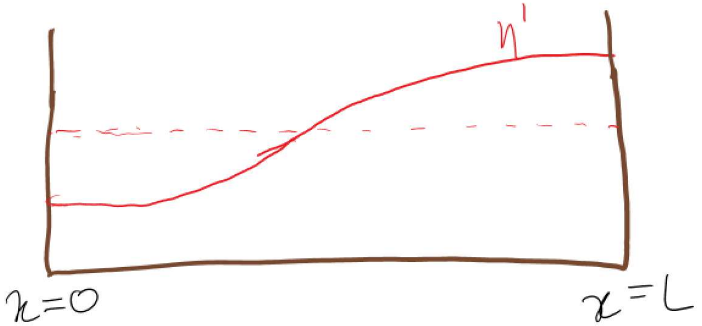

Thus: $i\omega\hat{u} + \frac{f^2}{i\omega}\hat{u} = -gH\frac{\partial\hat{\eta}'}{\partial x}$

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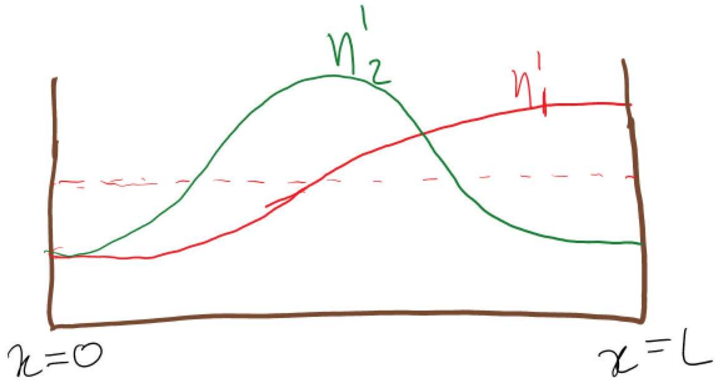
Let $\hat{\eta}' = \sum B_n \cos\left(\frac{n\pi x}{L}\right)$



Properties of the tides

How is the size of my ocean basin related to the amplitude of the tide????

Let $\hat{\eta}' = \sum B_n \cos\left(\frac{n\pi x}{L}\right)$

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$$\sum_n B_n \cos\left(\frac{n\pi x}{L}\right) - \frac{gH}{f^2-\omega^2}\frac{\partial^2}{\partial x^2} B_n \cos\left(\frac{n\pi x}{L}\right) = -\frac{4A}{C}\sum_n C_n \cos\left(\frac{n\pi x}{L}\right)$$

where $x = \sum_n C_n \cos\left(\frac{n\pi x}{L}\right)$

Properties of the tides

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where $x = \sum_n C_n \cos\left(\frac{n\pi x}{L}\right) \times \cos\left(\frac{m\pi x}{L}\right)\int_0^L dx$

Properties of the tides

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where $x = \sum_n C_n \cos\left(\frac{n\pi x}{L}\right) \times \cos\left(\frac{m\pi x}{L}\right)\int_0^L dx$

$$C_n = \frac{\int_0^L \cos\left(\frac{n\pi x}{L}\right) x \, dx}{\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx} = \begin{cases} 0 & n \text{ even} \\ \frac{2L}{n^2\pi^2} & n \text{ odd} \end{cases}$$

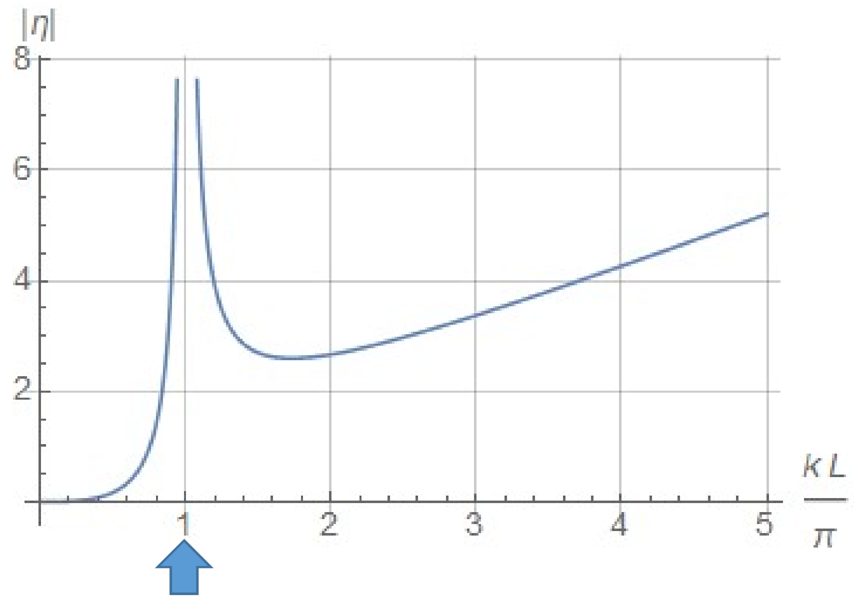
$$\hat{\eta}' = -\frac{4A}{C}\sum_{n=1,3,\ldots}^{\infty} \frac{2L}{n^2\pi^2} \frac{1}{1 - \frac{gHn^2\pi^2}{L^2(\omega^2-f^2)}} \cos\left(\frac{n\pi x}{L}\right)$$

Mode n=3 is ~10% of mode n=1… so ignore it.

Properties of the tides

How is the size of my ocean basin related to the amplitude of the tide????

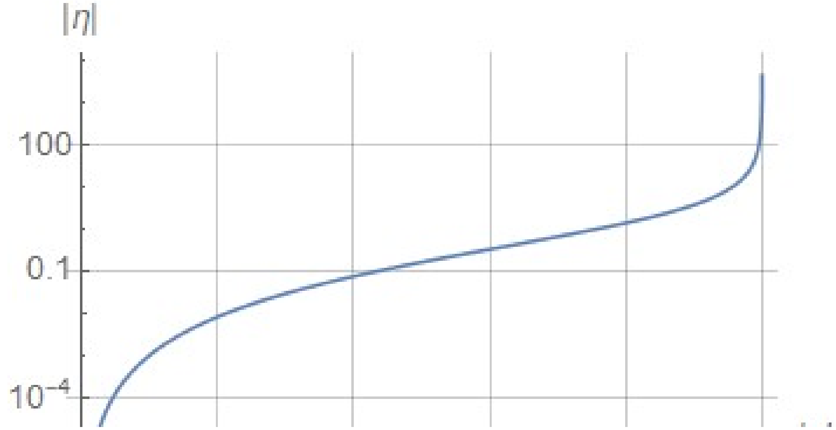
$$\hat{\eta}' \simeq -\frac{4A}{C}\,\frac{2L}{\pi^2}\,\frac{1}{1-\left(\frac{\pi}{KL}\right)^2}\cos\left(\frac{\pi x}{L}\right)$$



Here we have used the previous definition

$$\omega = \pm\sqrt{f^2 + K^2 gH}$$

where $\frac{1}{K} \sim$ 2000 km



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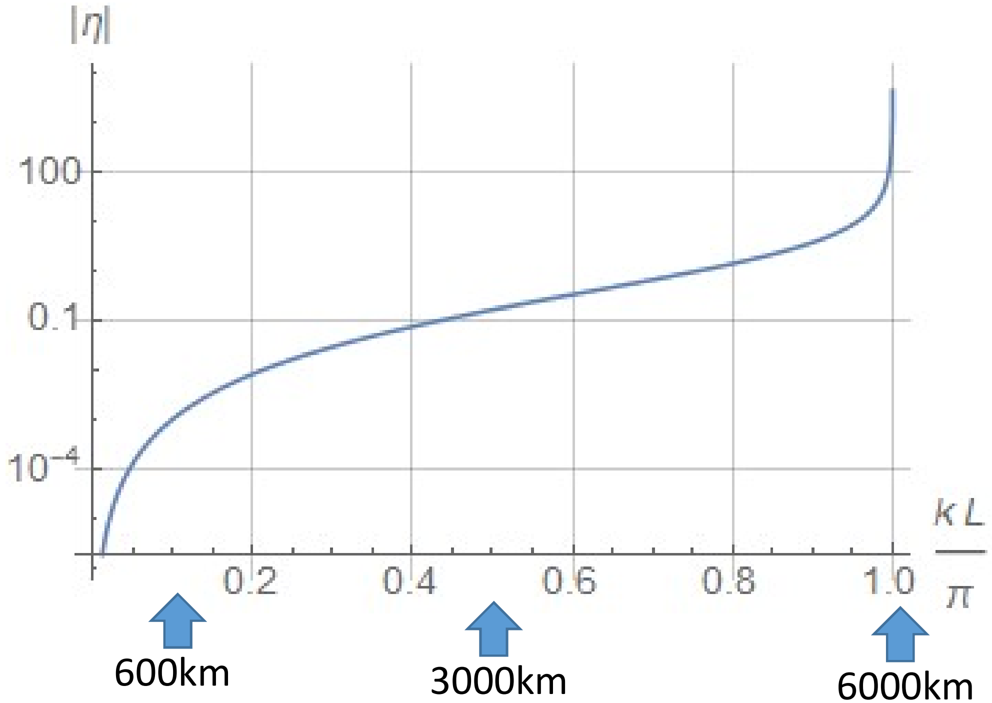

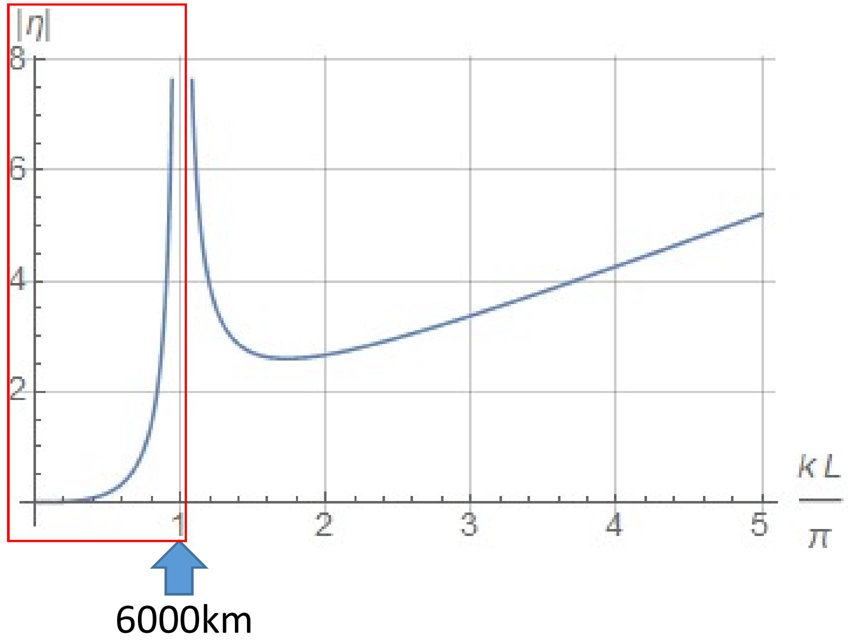

Properties of the tides

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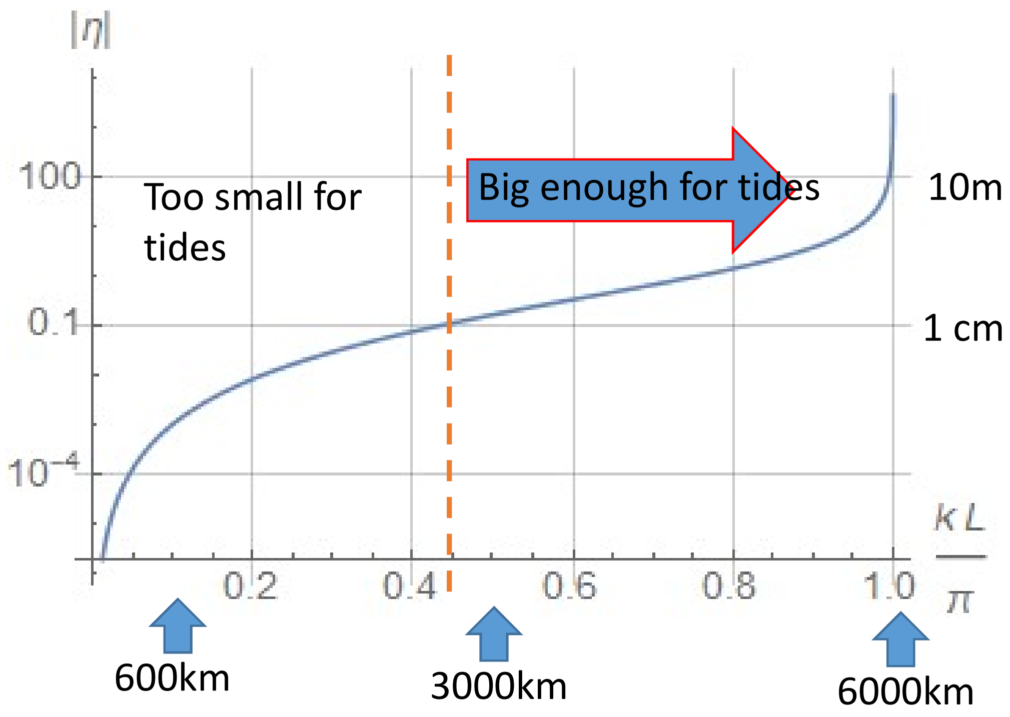
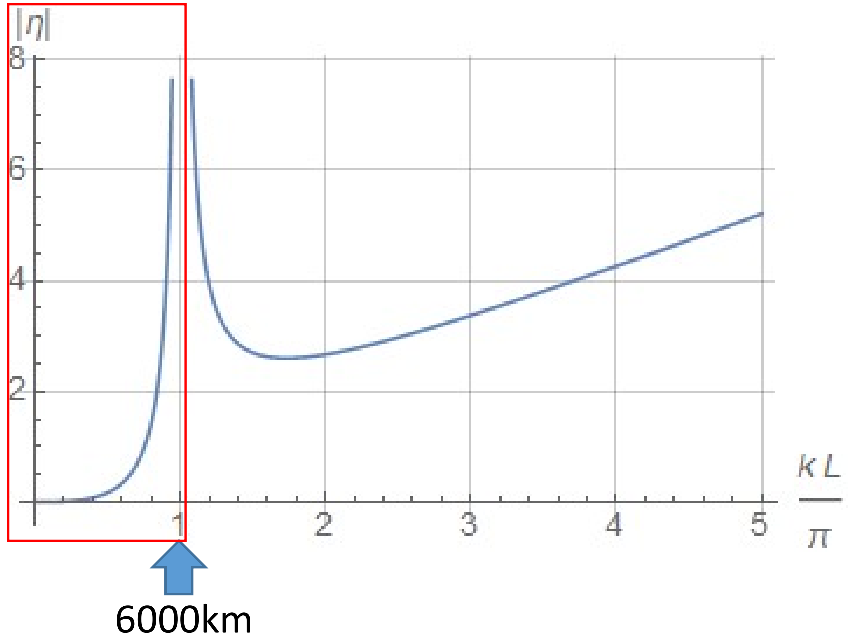
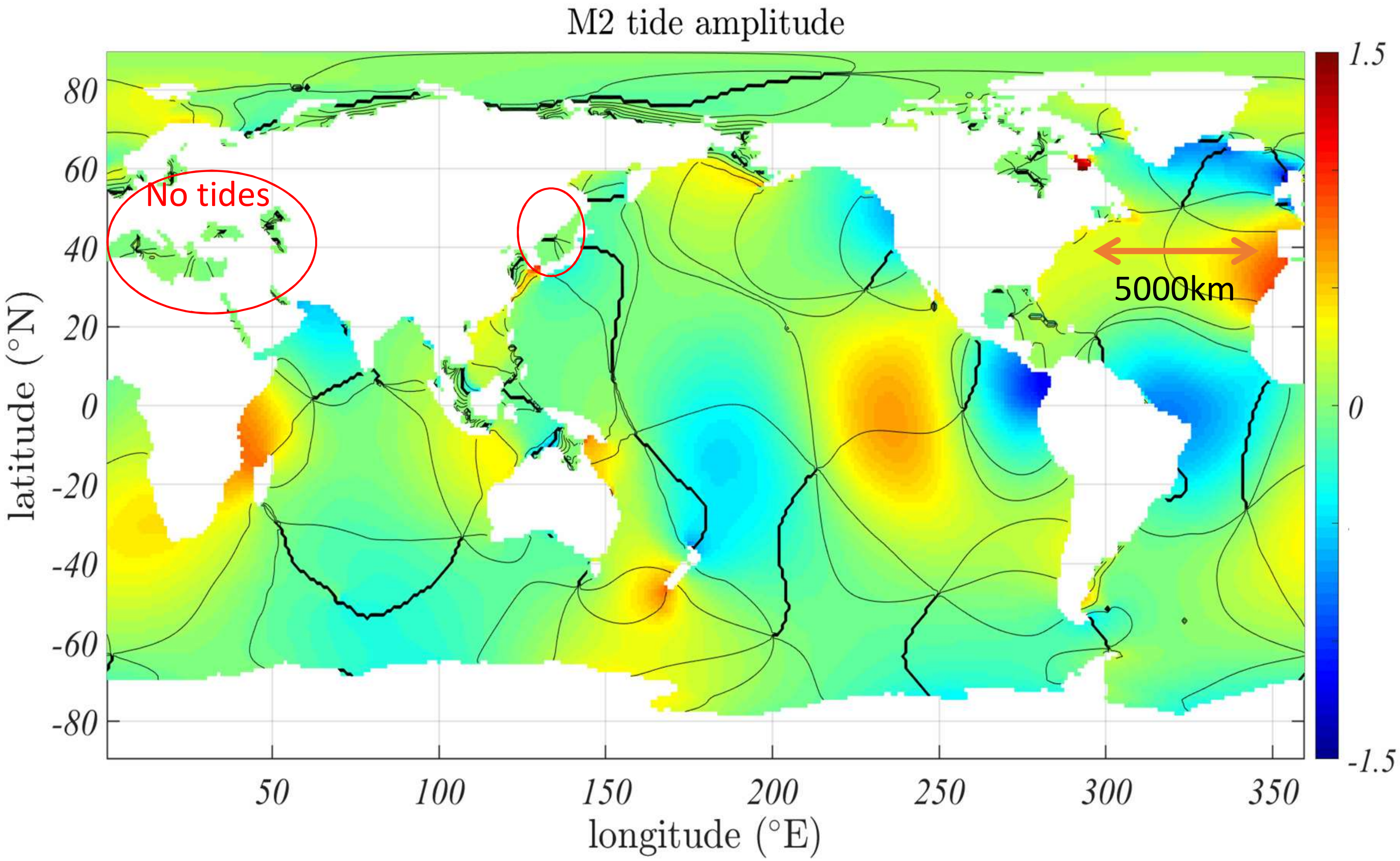
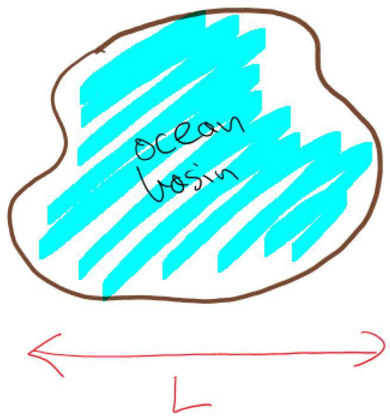




Comparison

How is the size of my ocean basin related to the amplitude of the tide????





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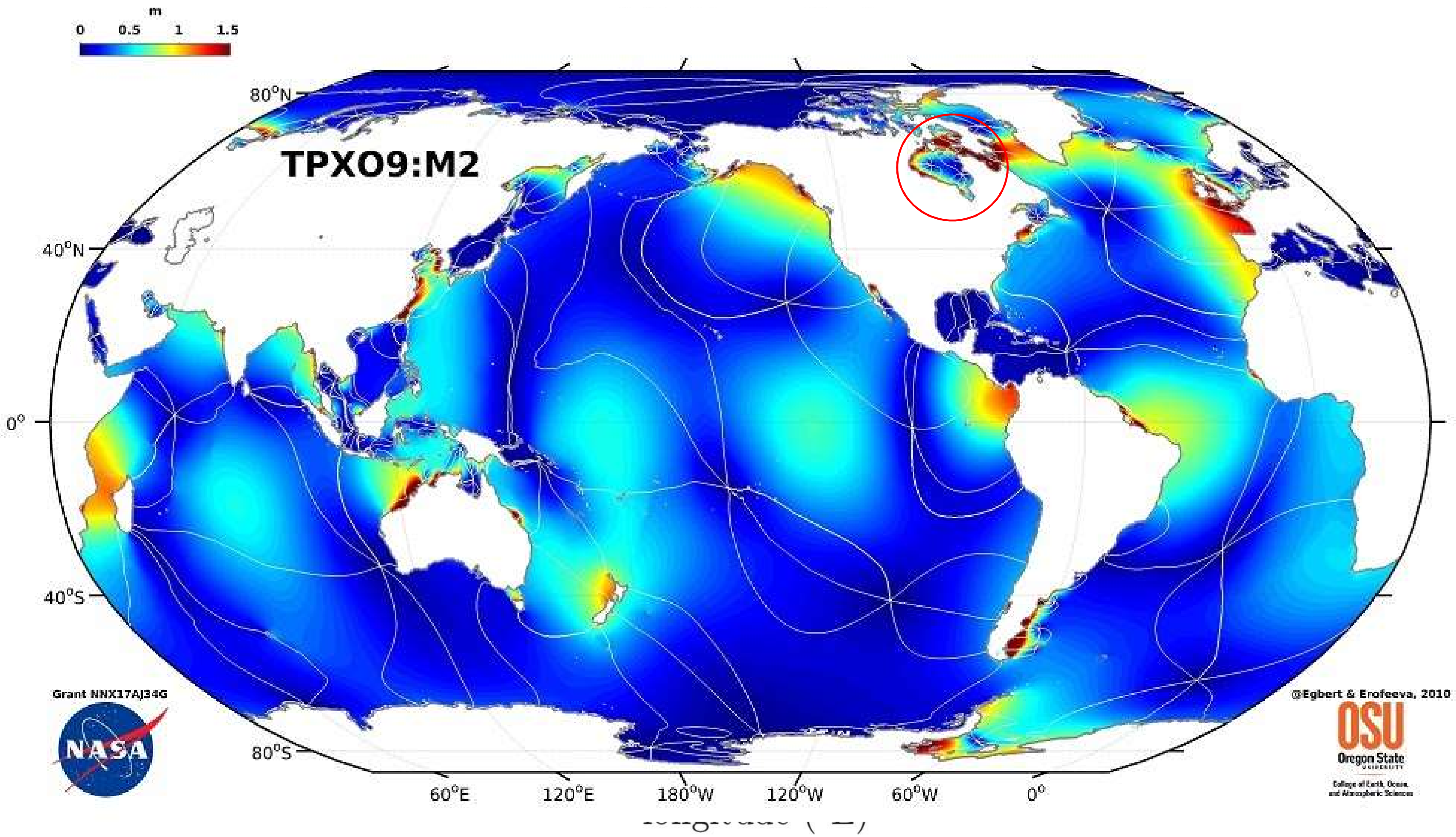


Hudson Bay
270 m deep max

L>500km for tides

https://www.tpxo.net/global

Comparison with Atlantic basin over time

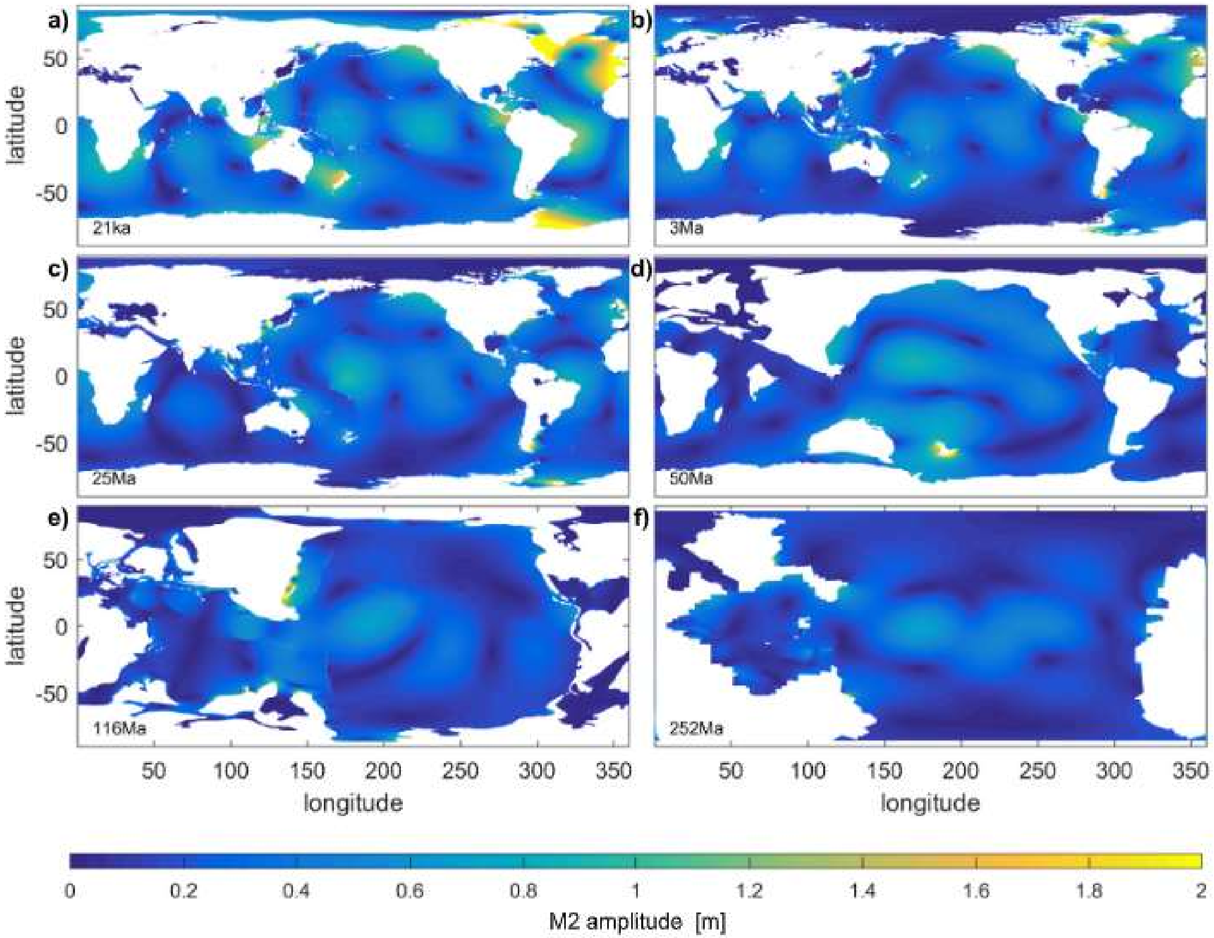
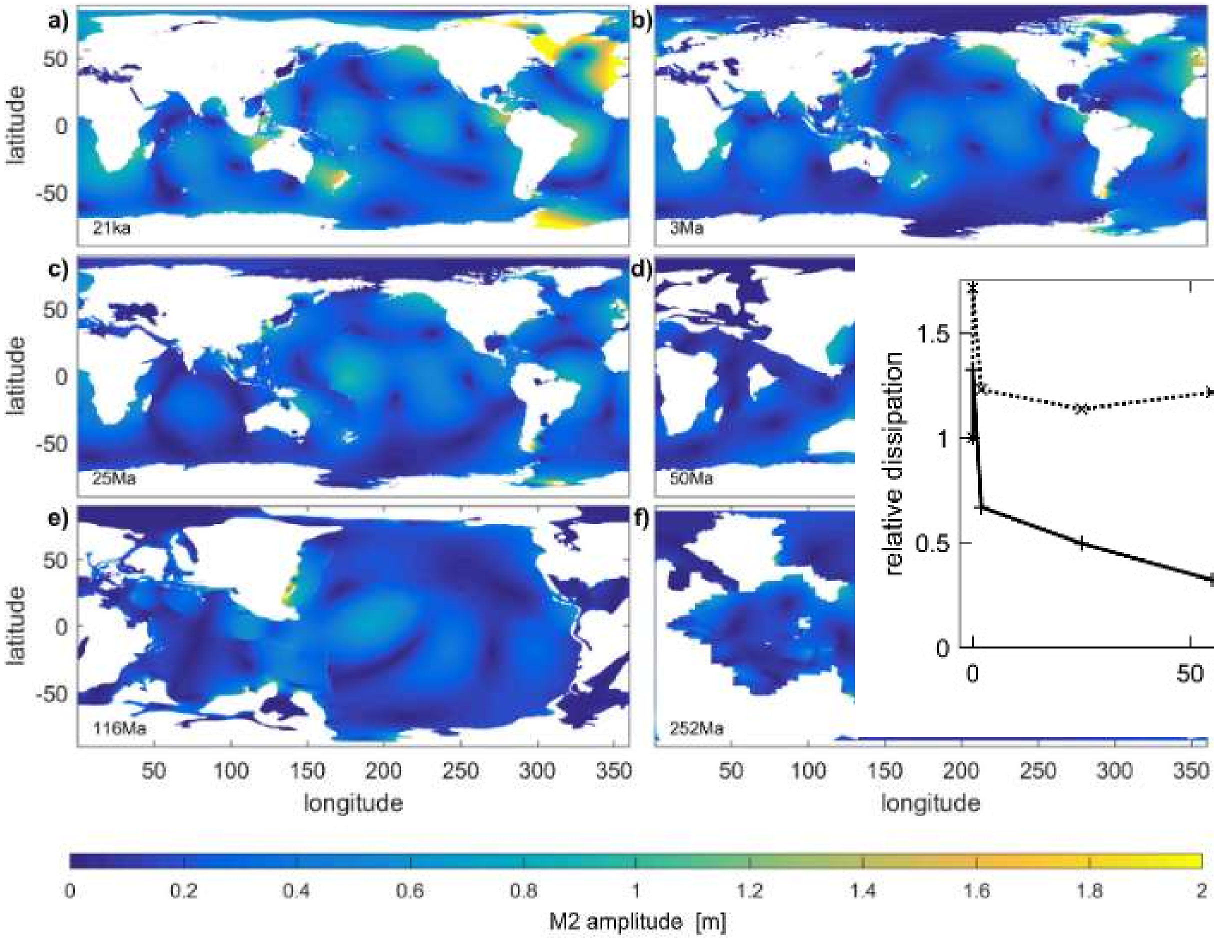


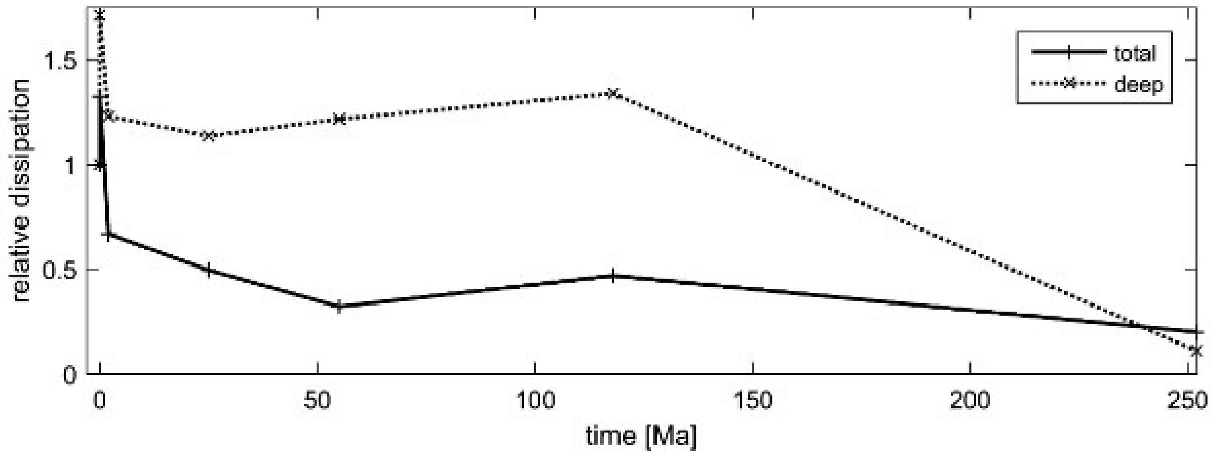
Fig. 2. Shown are the M_2 tidal amplitudes for the LGM (a), Pliocene (b), Miocene (c), Eocene (d), Cretaceous (e) and Permian–Triassic (f).

From Green et al., 2017

Comparison with Atlantic basin over time

Stronger tides = more energy dissipation = more ocean mixing!





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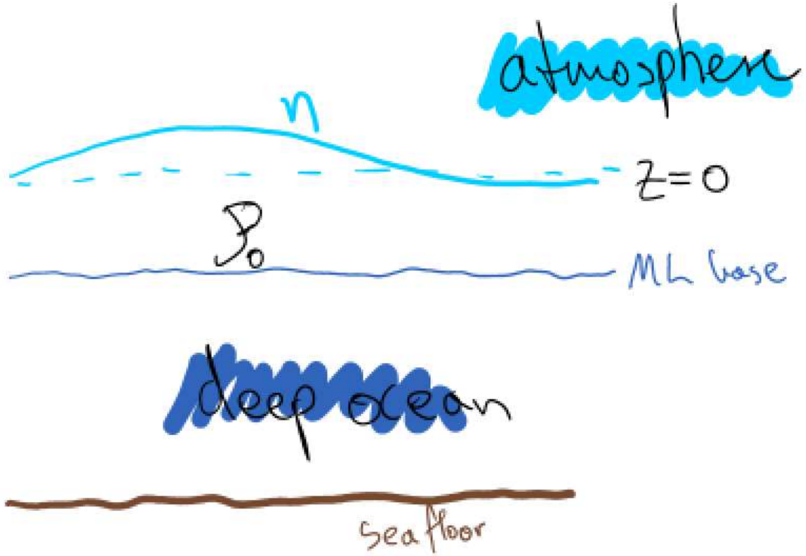
BGWs versus surface gravity waves (SGWs)



BGWs versus surface gravity waves (SGWs)

Kinematic boundary condition

Surface equations

$$w = \frac{D\eta}{Dt}$$

Uniform density, **non-hydrostatic** Navier Stokes

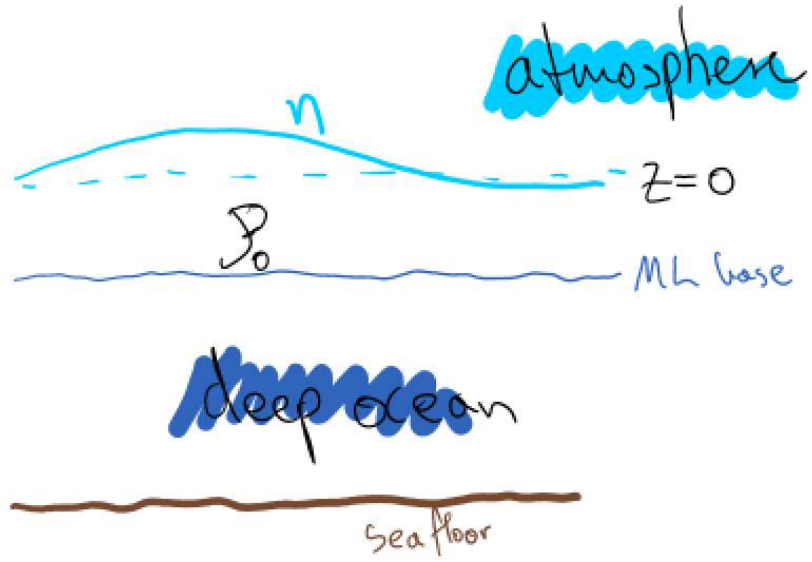
Interior equations

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g$$

$$\nabla \cdot \underline{u} = 0.$$



BGWs versus surface gravity waves (SGWs)

Linearised surface
(z=0)

$$w = \frac{\partial \eta}{\partial t}$$

Linearised interior

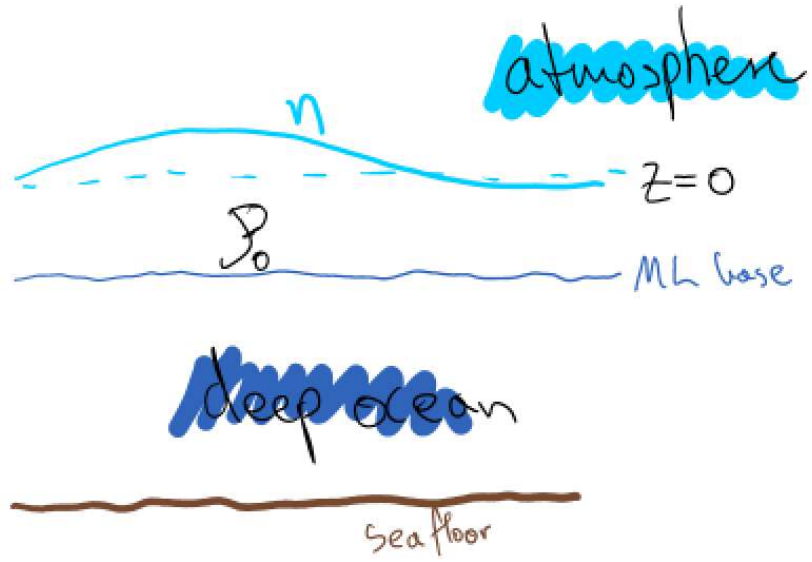
Let: $p = \rho_0 g z + p'$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$



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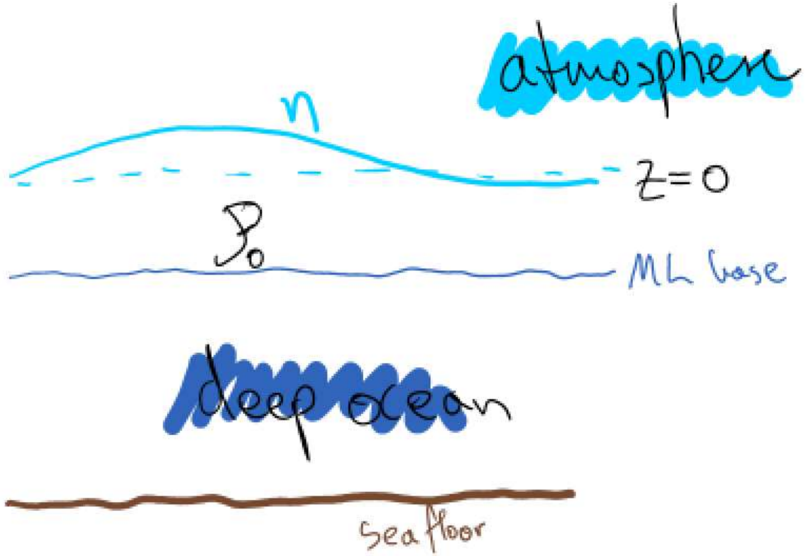
$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}\Big|_{z=0}$$

$$\frac{\partial}{\partial t}\left(\nabla^2 p'\right) = 0$$



BGWs versus surface gravity waves (SGWs)

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(z=0)

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$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial w}{\partial t}$$
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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

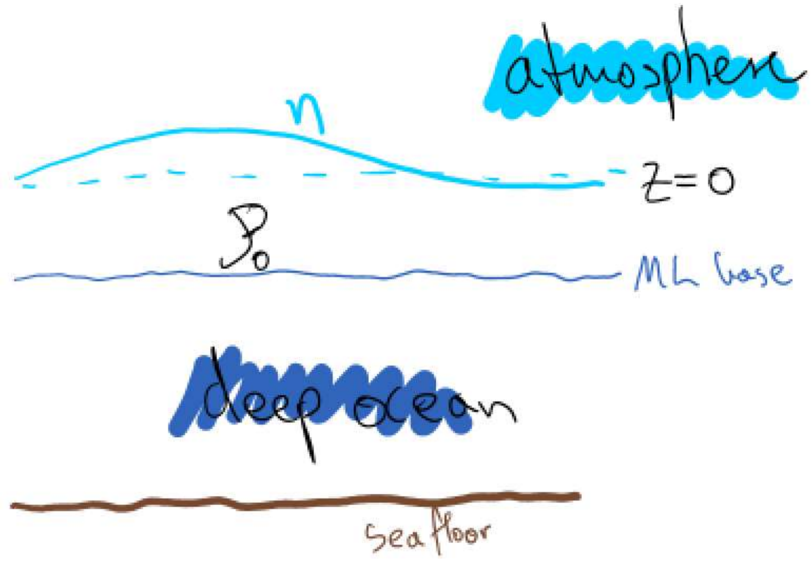
$$\frac{\partial}{\partial t}\left(\nabla^2 p'\right) = 0$$

Let $p' \sim \hat{p}' e^{i(kx+ly-\omega t)}$

Then $\left(-(k^2+l^2) + \frac{\partial^2}{\partial z^2}\right)\hat{p}' = 0$

$$\boxed{\Rightarrow \hat{p}' \simeq \hat{p}_0' e^{-Kz} \quad \text{where } K = \sqrt{k^2+l^2}}$$

Wave amplitude decays with depth, according to horizontal wavelength



BGWs versus surface gravity waves (SGWs)

Linearised surface (z=0)

$$w = \frac{\partial \eta}{\partial t}$$

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}\bigg|_{z=0}$$

Linearised interior

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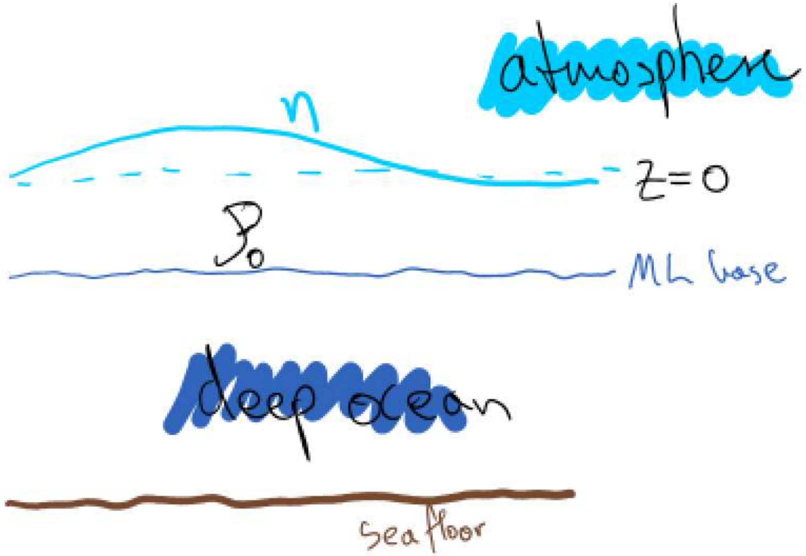
$\Rightarrow \hat{p}' = \hat{p}'_0 e^{-Kz}$

where $K = \sqrt{k^2+l^2}$



At the surface the total pressure must be constant to match atmospheric pressure, thus

$p = p' + \rho_0 g \eta = 0$ at surface

$\Rightarrow \hat{p}'_0 = -\rho_0 g \eta$

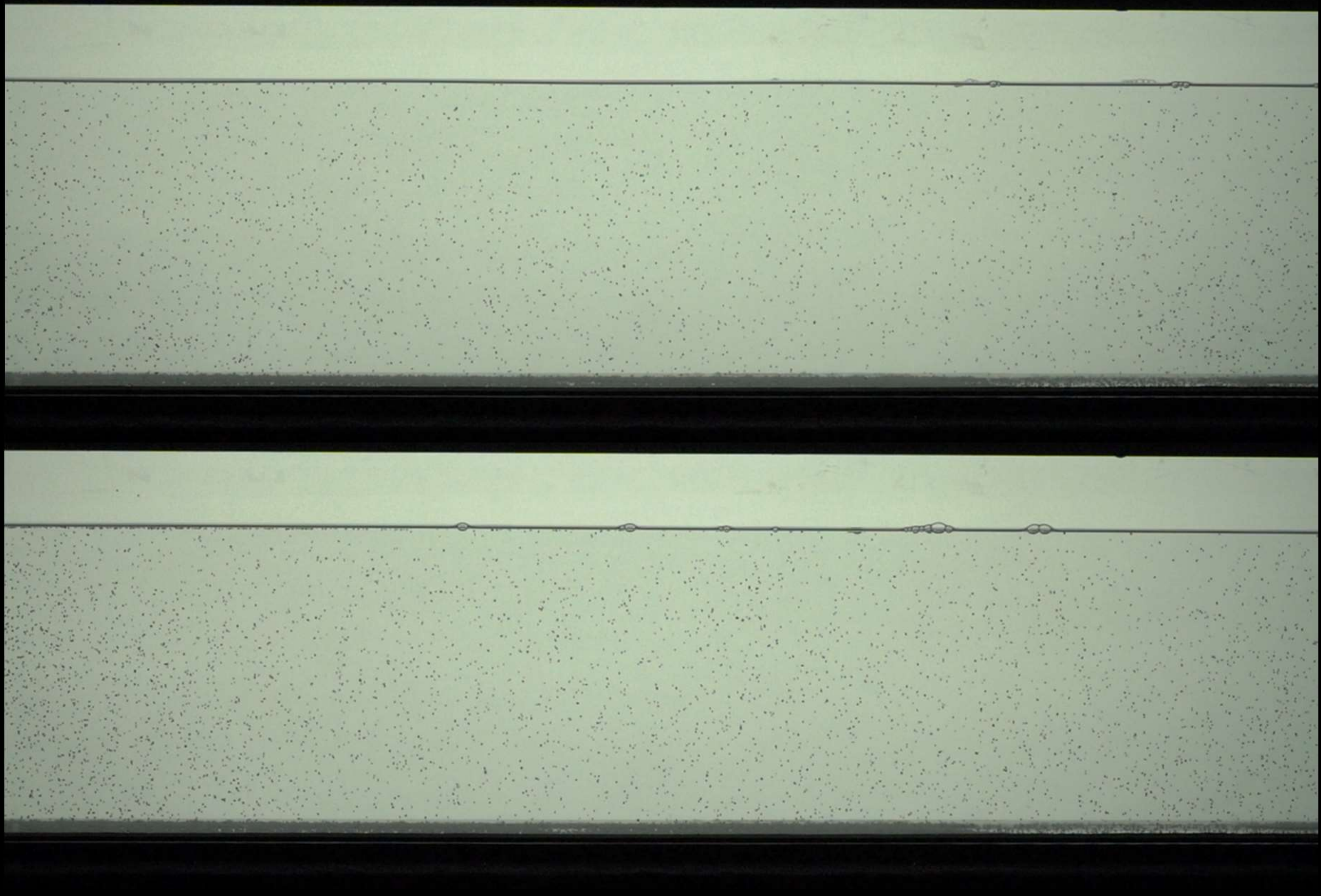
$\Rightarrow \frac{\partial \hat{p}'}{\partial z}\Big|_{z=0} = \rho_0 g K \eta$

$\Rightarrow \frac{\partial^2 \eta}{\partial t^2} = -gK\eta$

$\Rightarrow \boxed{\omega^2 = gK}$ Dispersion relation

Surface gravity waves (SGWs)

$$\omega = \sqrt{gK}$$
$$P \sim \rho_0 g\, \eta\, e^{-K\,z}$$



SGWs

$$\omega = \pm\sqrt{gK}$$

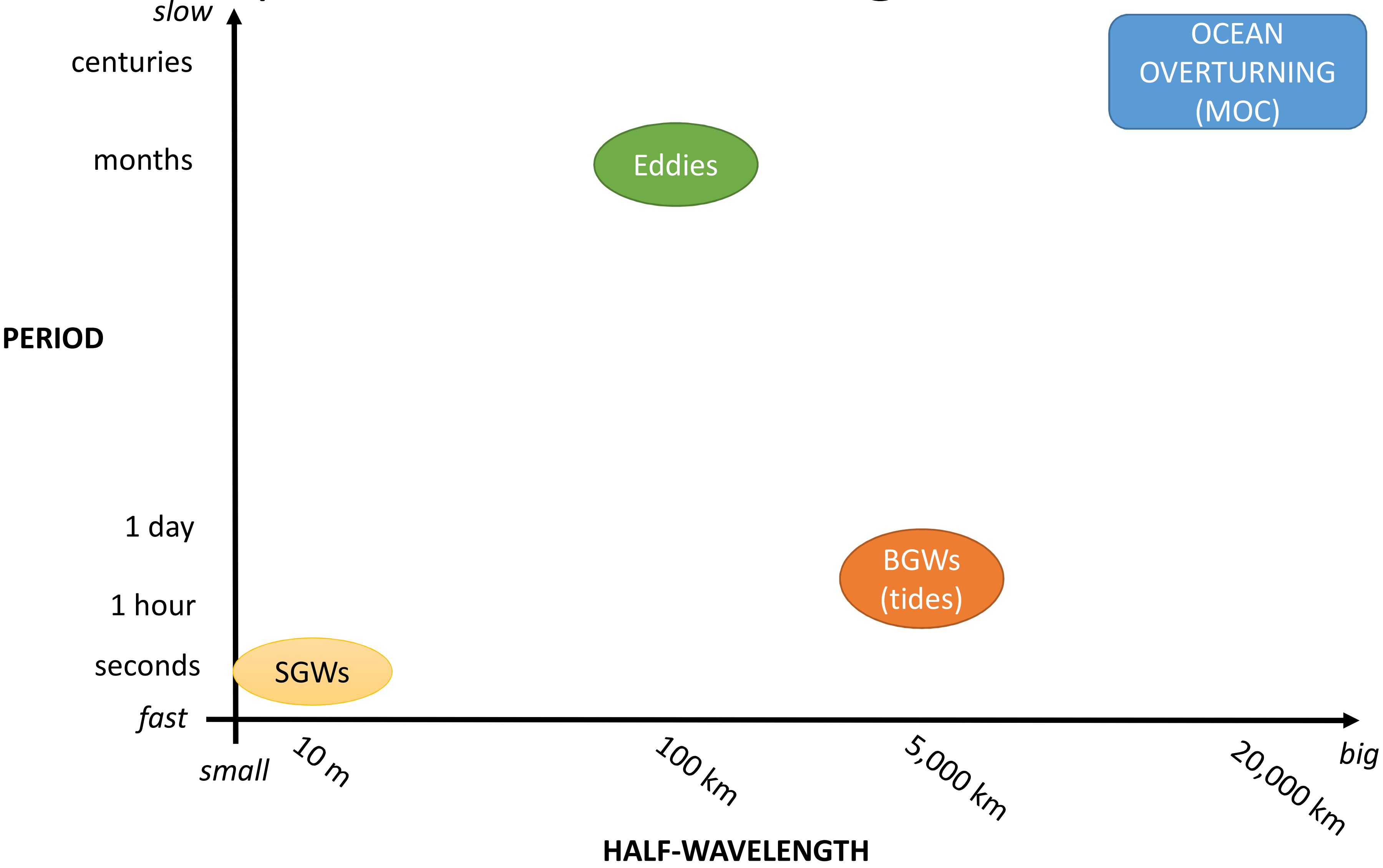
- Generated by perturbations at the surface (wind!)
- $\frac{2\pi}{\omega} \sim 3\ seconds\ \Rightarrow \frac{2\pi}{k} \sim 14\ m$
- Small spatial scales
- Non-hydrostatic = large vertical acceleration
- Shallow, surface intensified motion

BGWs

$$\omega = \pm\sqrt{f^2 + K^2 gH}$$

- Generated by full depth/body force perturbations (tidal forces)
- $\frac{2\pi}{\omega} \sim 12.4\ hrs \Rightarrow \frac{2\pi}{k} \sim 10{,}000 km$
- Large spatial scale
- Hydrostatic = slow vertical acceleration
- Full-depth barotropic motion

Ocean space-time scale diagram



Some references

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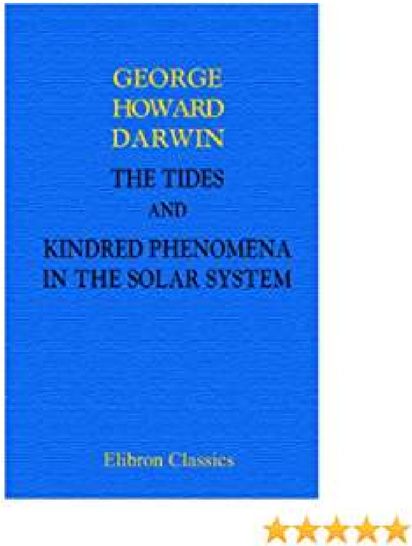
  - Acheson, David J. "Elementary fluid dynamics." (1991)



- Paleo-tides
  - Green. "Ocean tides and resonance." *Ocean Dynamics* 60.5 (2010): 1243-1253.
  - Green, J. A. M., et al. "Explicitly modelled deep-time tidal dissipation and its implication for Lunar history." *Earth and Planetary Science Letters* 461 (2017): 46-53.

Spherical coordinates: water world

$$\nabla_h \cdot \mathbf{A} = \frac{1}{r\cos\theta}\frac{\partial}{\partial\phi}A_\phi + \frac{1}{r\cos\theta}\frac{\partial}{\partial\theta}A_\theta\cos\theta$$

$$\nabla_h A = \left(\frac{1}{r\cos\theta}\frac{\partial A}{\partial\phi}, \frac{1}{r}\frac{\partial A}{\partial\theta}\right)$$

$$f = 2\Omega\sin\theta$$