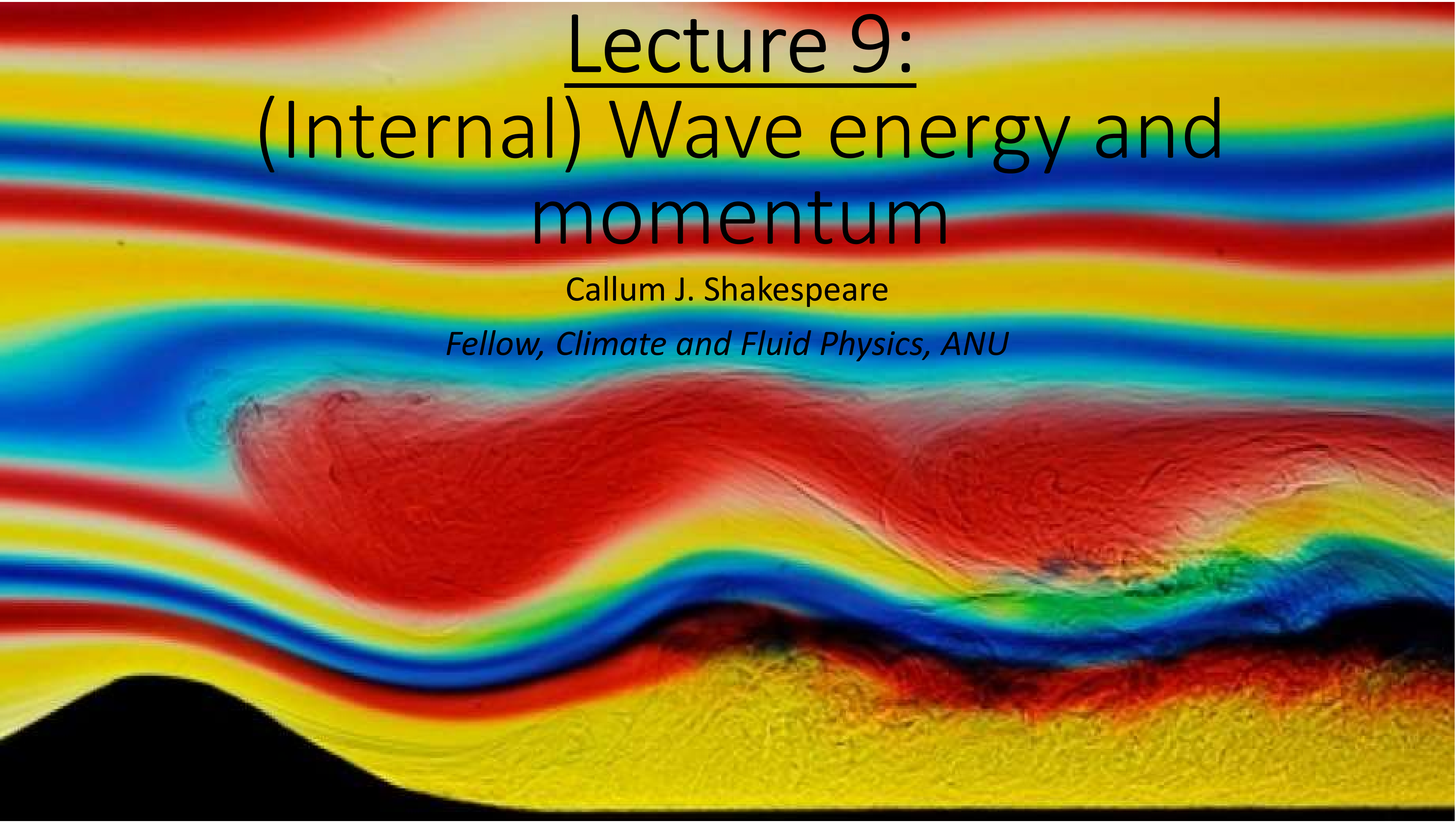
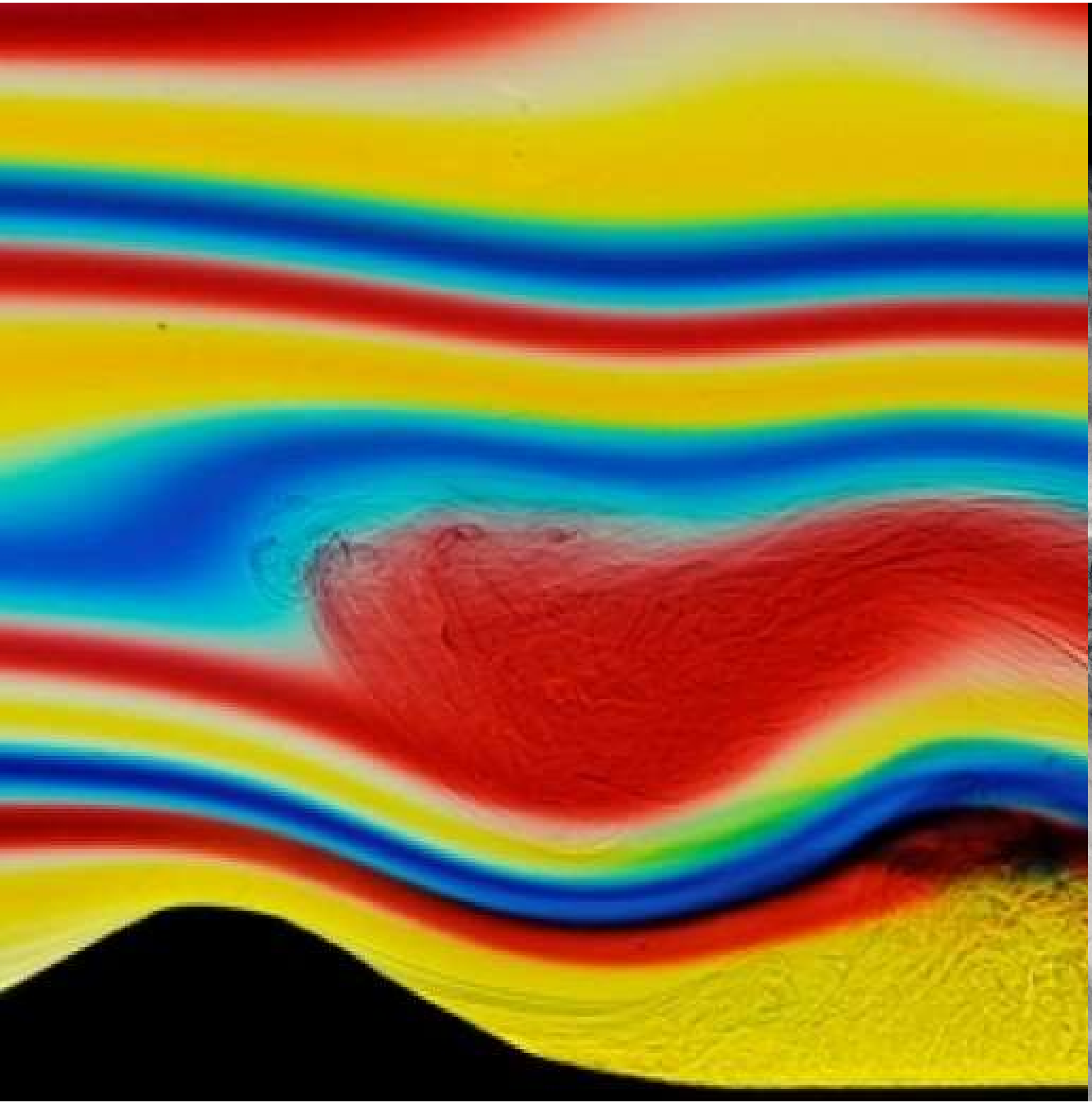
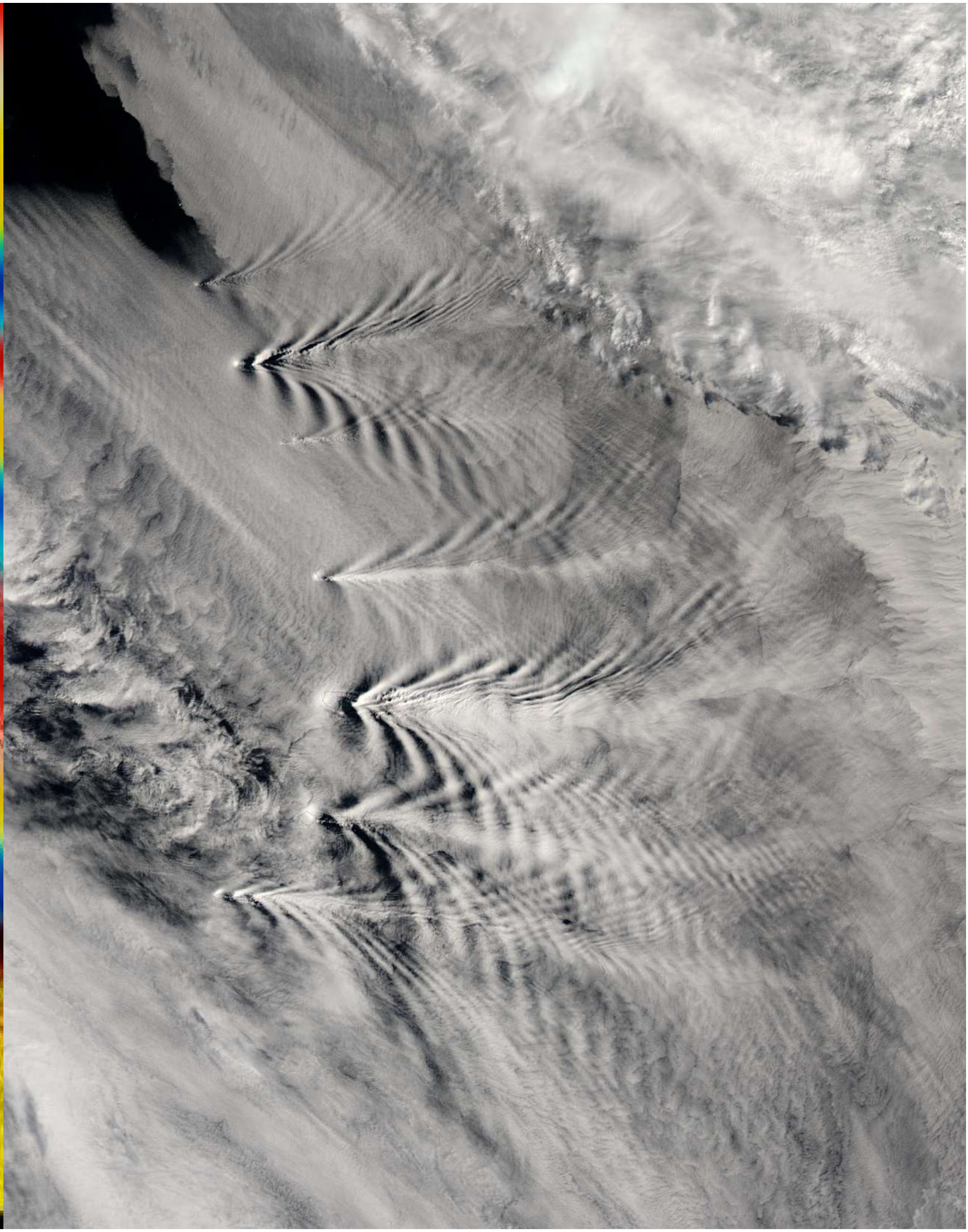


Lecture 9:
(Internal) Wave energy and momentum

Callum J. Shakespeare

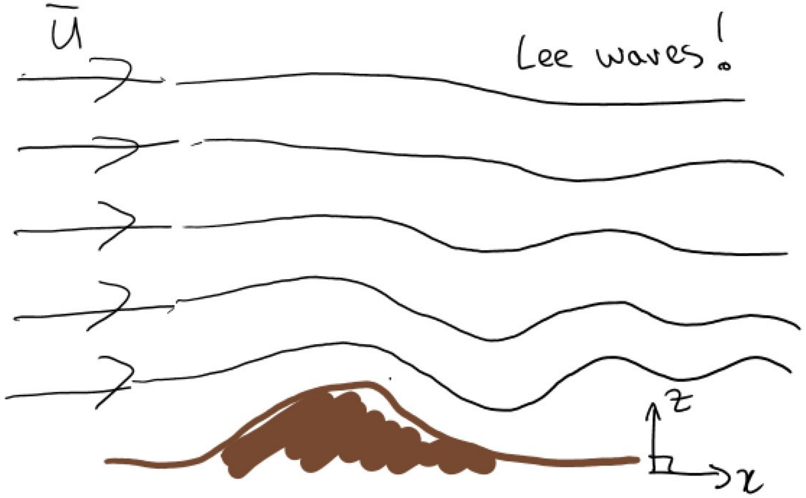
Fellow, Climate and Fluid Physics, ANU



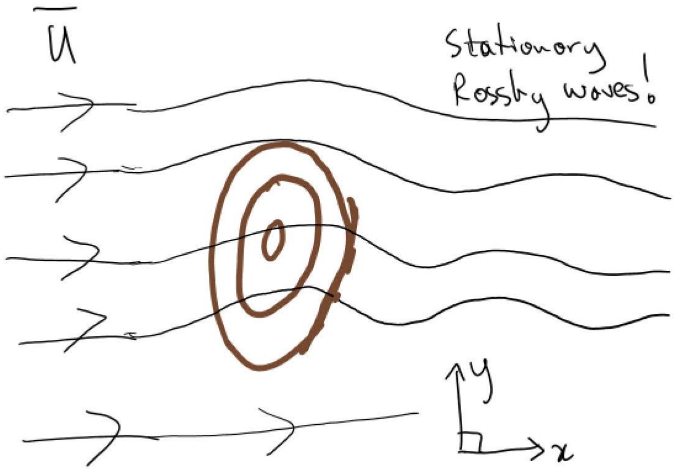


The mechanism behind the waves (Lecture 7)

Gravity waves

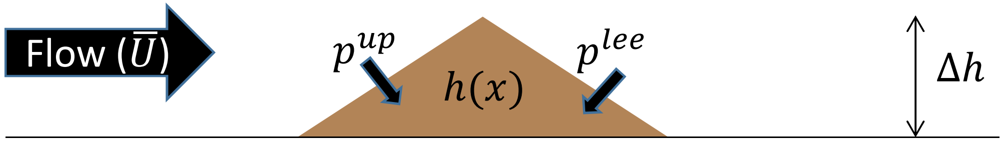


Rossby waves



- The bathymetry induces a z (or y) velocity to the flow, which reduces its buoyancy (or vorticity)
- If the perturbation is slow/weak, the flow remains in/near balance and returns to its original course….
- But if the perturbation is **fast/strong** ($\overline{U} \sim \frac{\omega}{k}$), it kicks of an oscillation in the lee of the obstacle…

Lee waves as an example



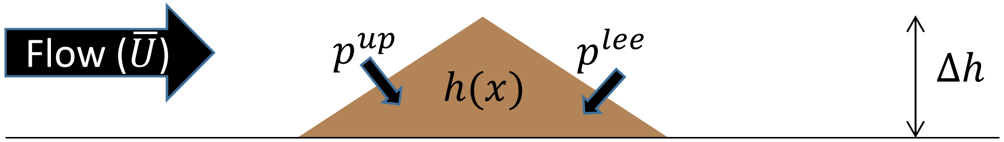
Pressure is the force per unit area
The horizontal force/stress on the hill is thus:

$$\tau = p^{up}\,\Delta h\; - p^{lee}\Delta h = \int p'\frac{dh}{dx}\,dx$$

The work done on the hill is

$$W = \tau \cdot \bar{U} = \int p'\bar{U}\frac{dh}{dx}\,dx$$

Lee waves as an example



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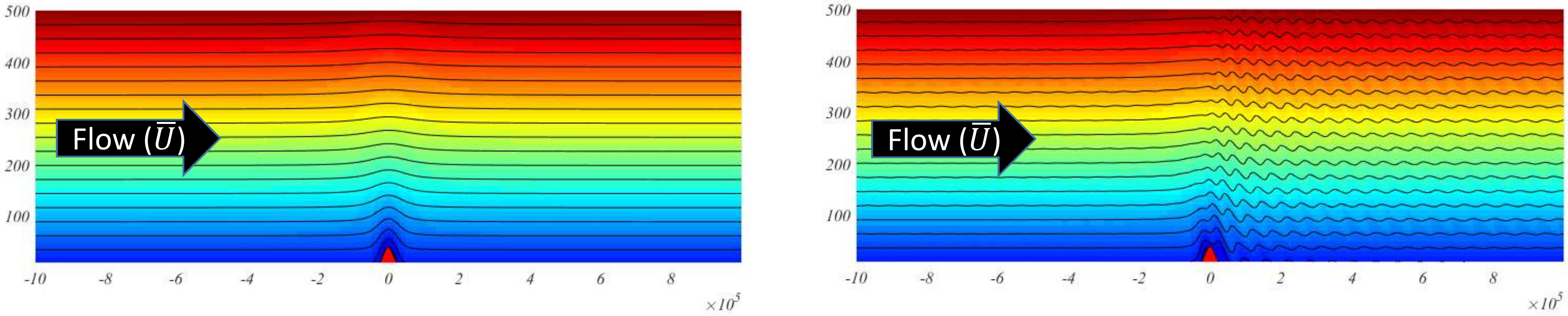
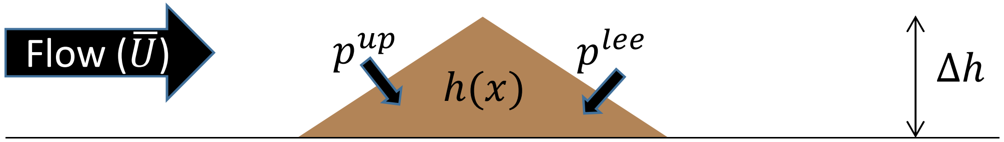


Lee waves as an example



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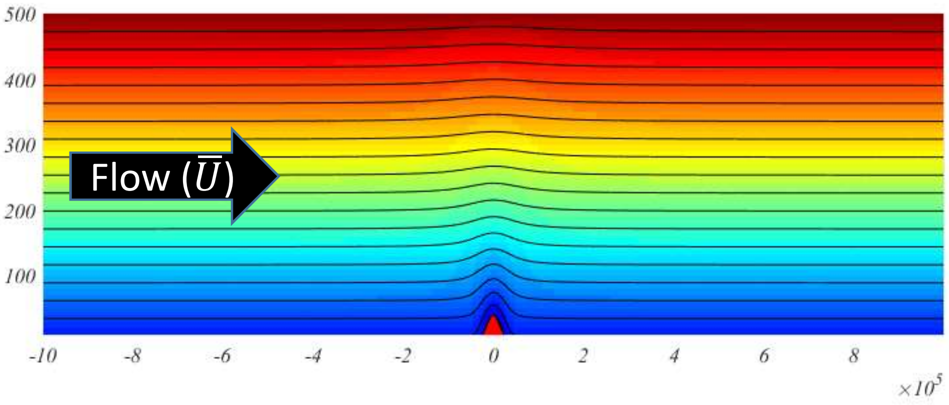
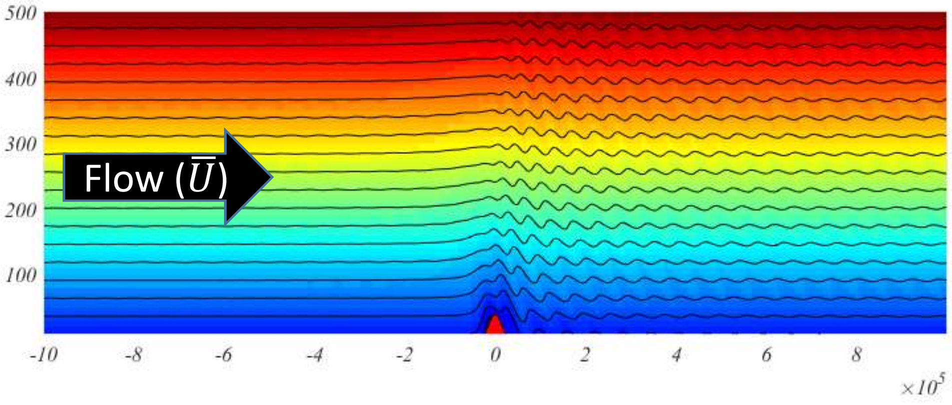
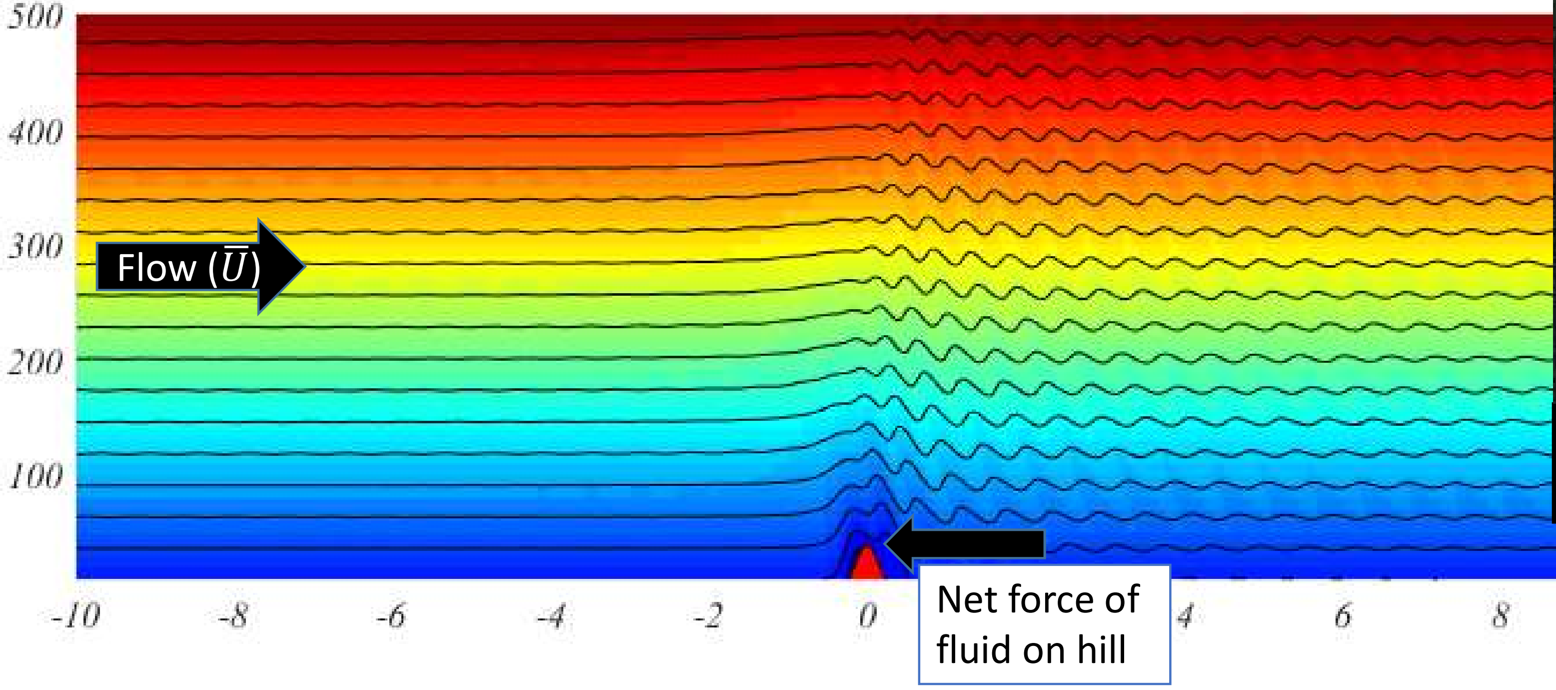
In asymmetric (wave) cases, we are extracting momentum and energy from the hill, into the flow.

The energy and momentum is carried by the waves





Lee waves as an example



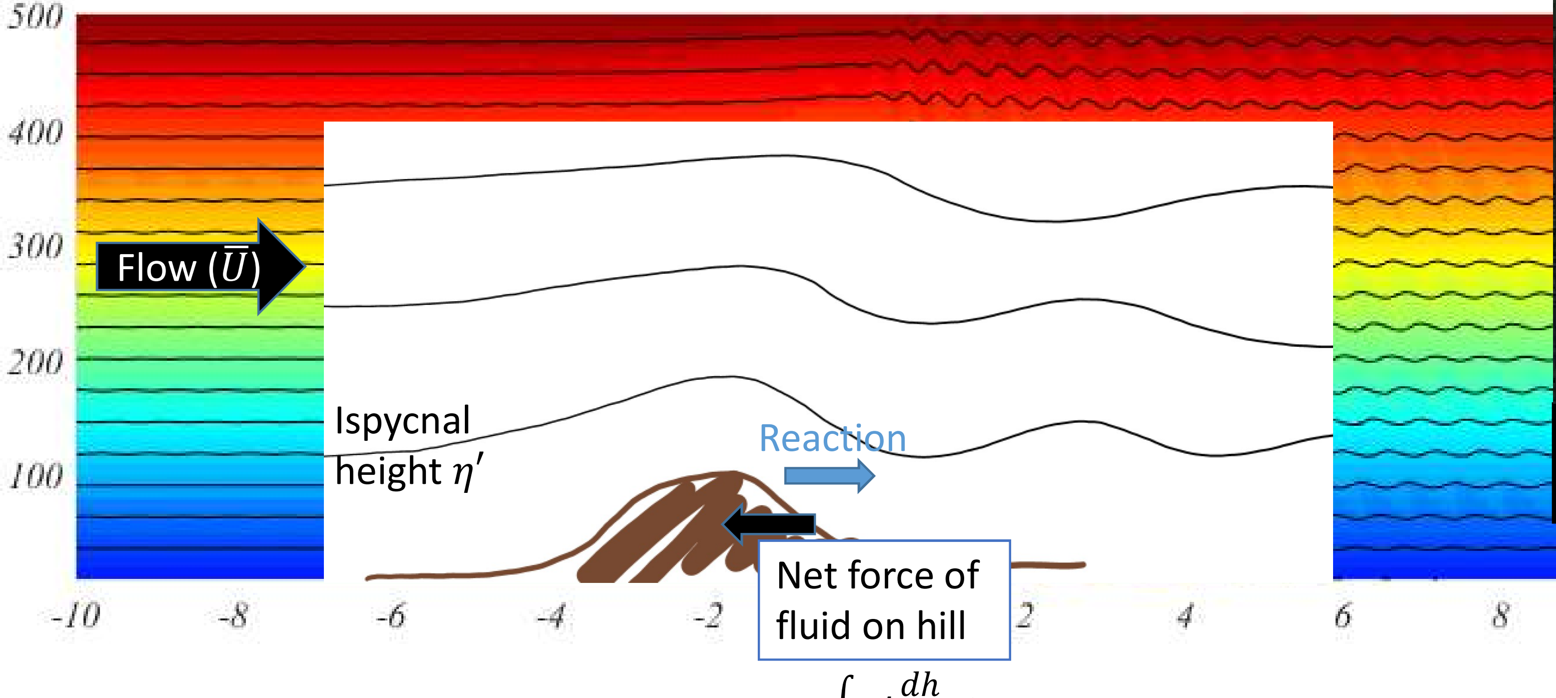
Flow ($\bar{U}$)

Net force of fluid on hill

For every action, there is an equal and opposite reaction

Where is the reaction force applied??????

Lee waves as an example



Flow ($\bar{U}$)

Ispycnal height η'

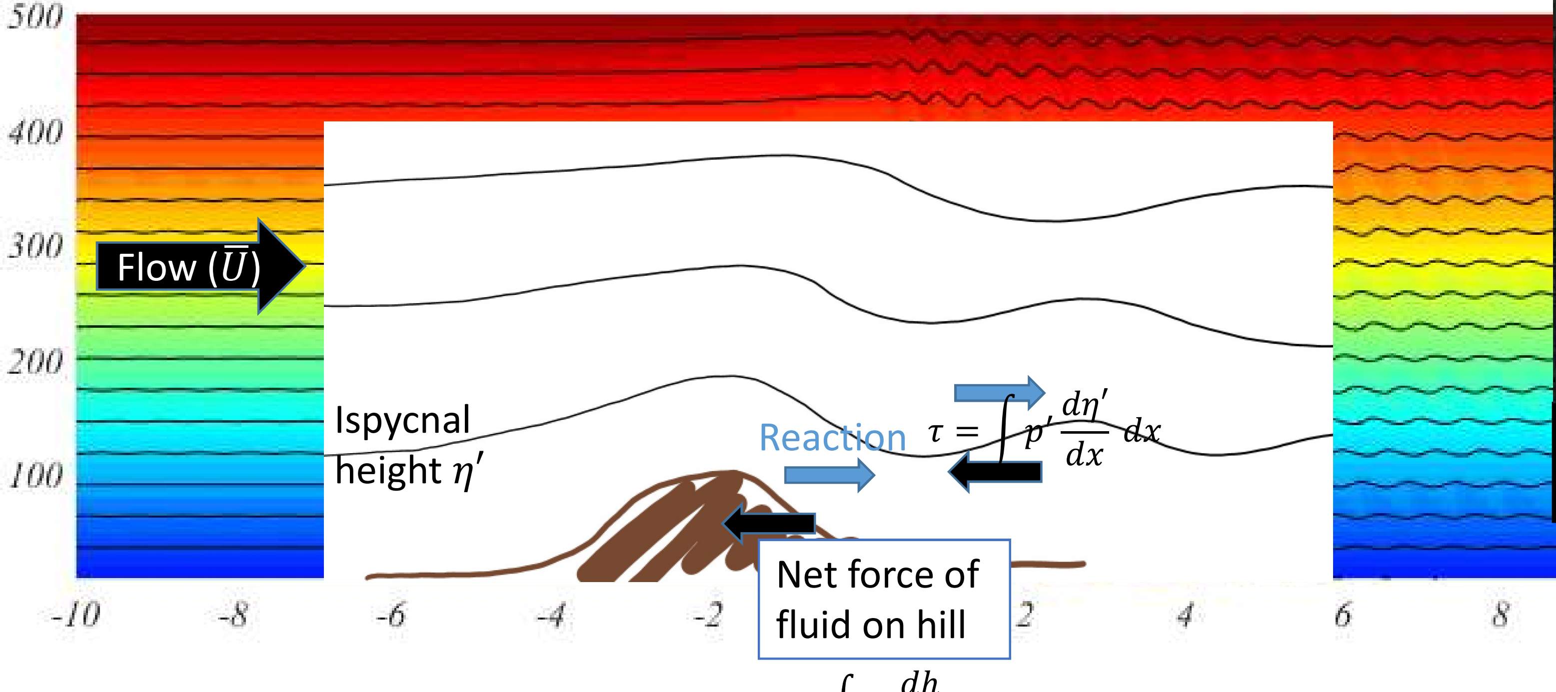
Reaction

Net force of fluid on hill

$$\tau = \int p' \frac{dh}{dx}\, dx$$

For every action, there is an equal and opposite reaction

Lee waves as an example



Flow ($\bar{U}$)

Ispycnal height η'

Reaction

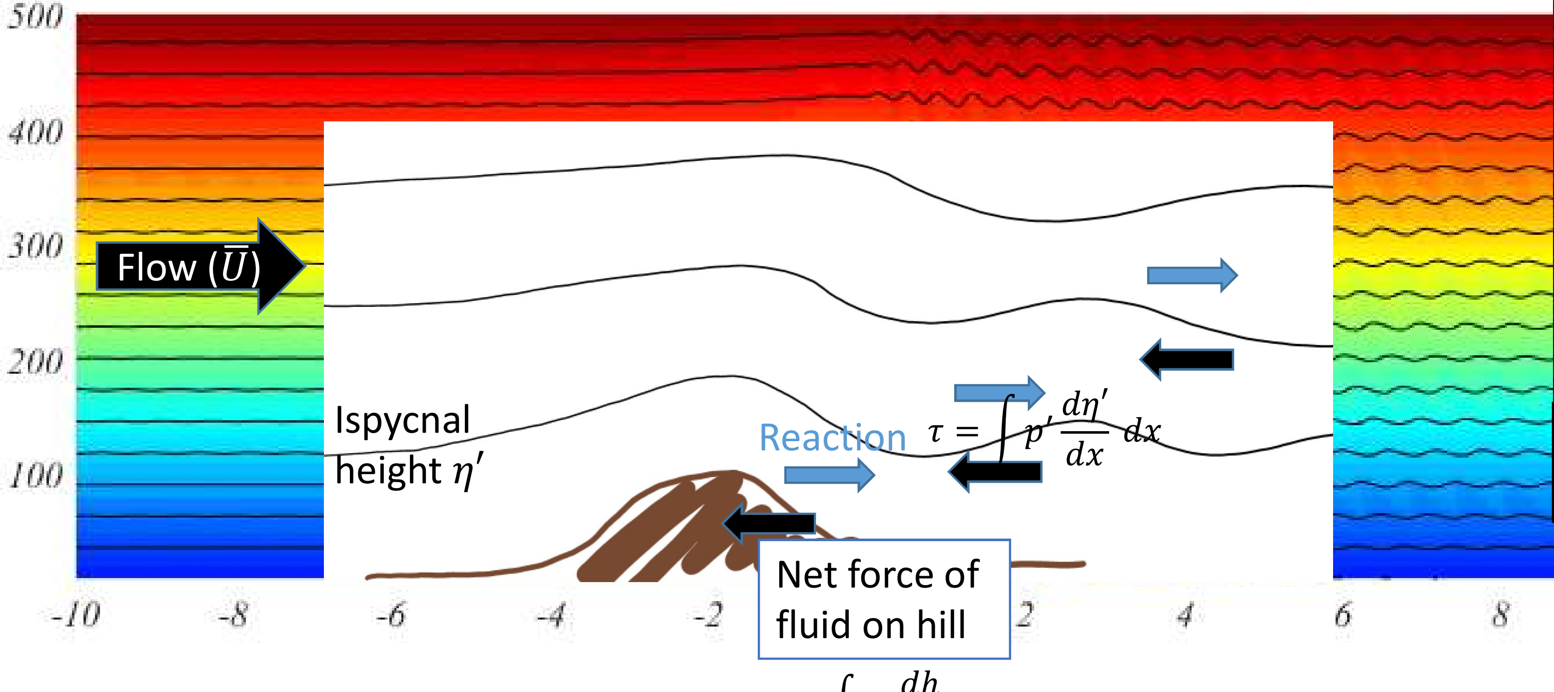
$$\tau = \int p' \frac{d\eta'}{dx}\, dx$$

Net force of fluid on hill

$$\tau = \int p' \frac{dh}{dx}\, dx$$

For every action, there is an equal and opposite reaction

Lee waves as an example



$$\tau = \int p' \frac{d\eta'}{dx}\, dx$$

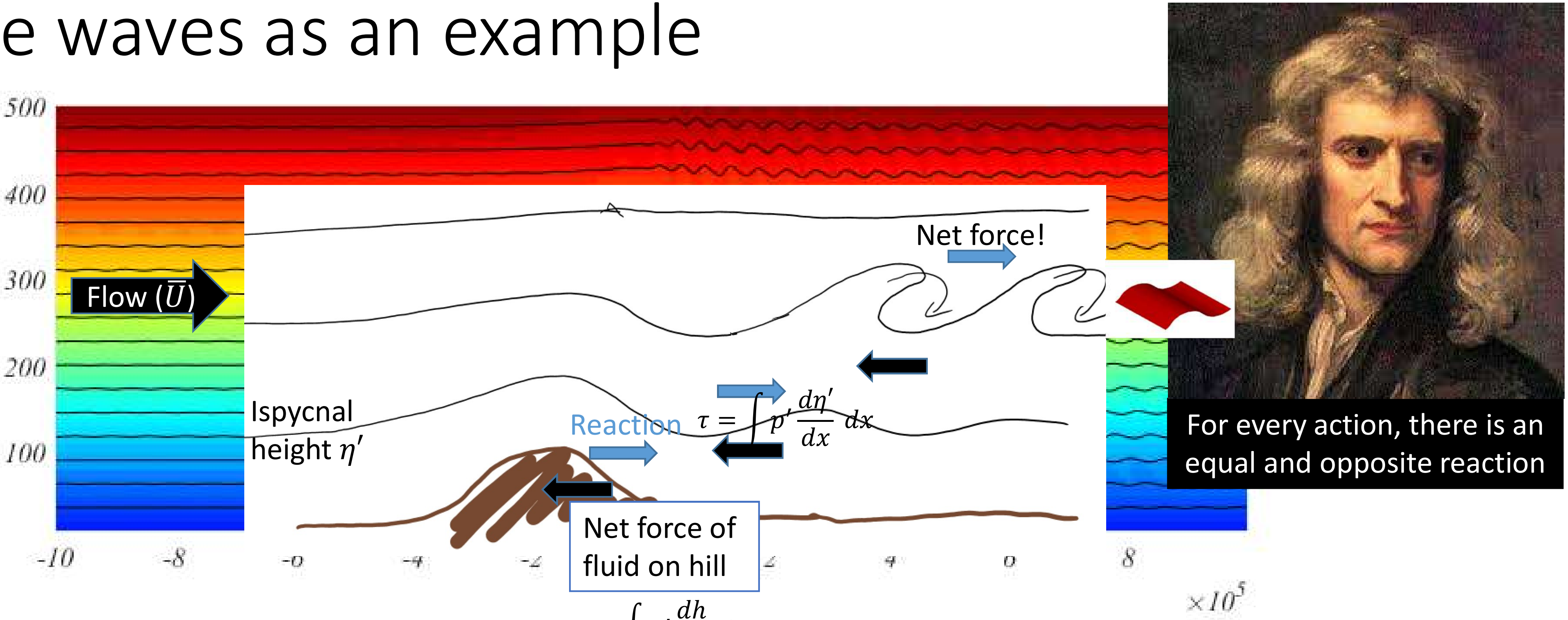
$$\tau = \int p' \frac{dh}{dx}\, dx$$

For every action, there is an equal and opposite reaction

We still haven’t found the NET reaction force!

The forces on the fluid are in balance at every level….

Lee waves as an example

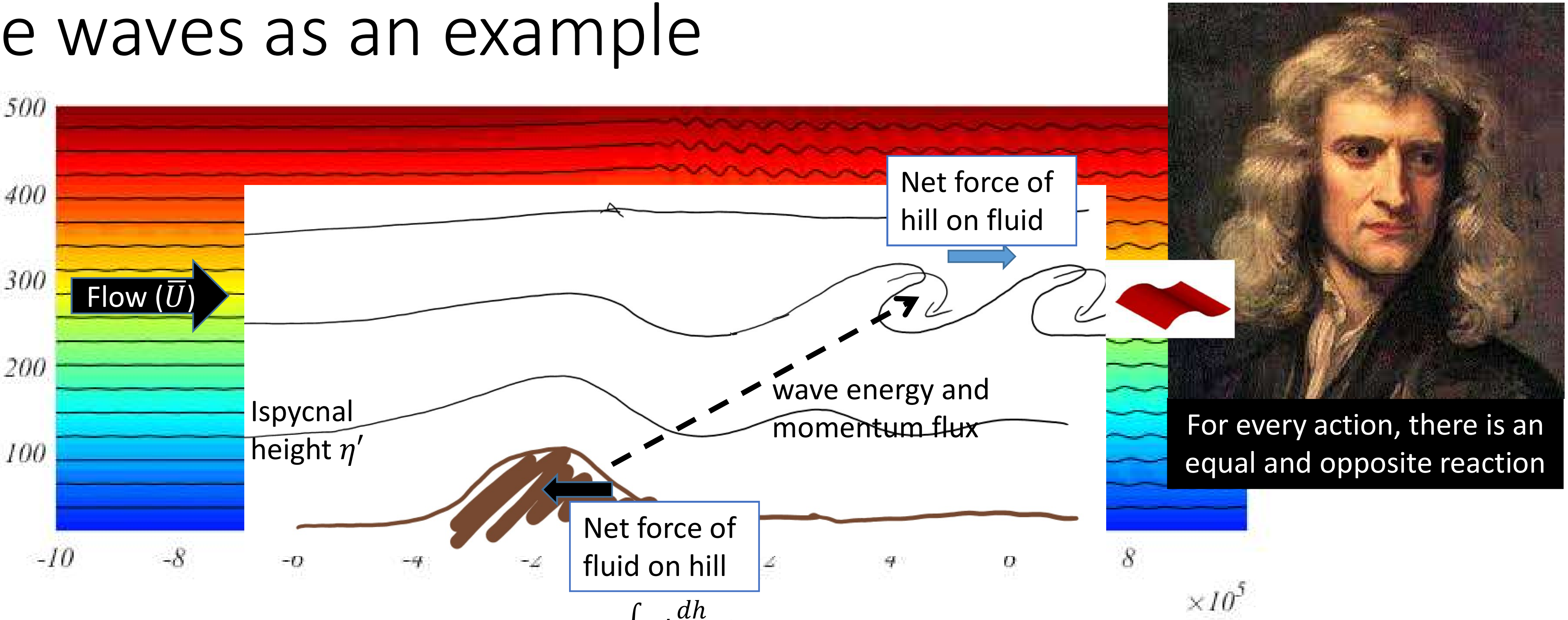


For every action, there is an equal and opposite reaction

$$\tau = \int p' \frac{dh}{dx}\, dx$$

A NET reaction force is only felt in the layer where the wave decays/attenuates

Lee waves as an example



For every action, there is an equal and opposite reaction

$$\tau = \int p' \frac{dh}{dx}\, dx$$

- A NET reaction force is only felt in the layer where the wave decays/attenuates
- This could be a LONG way from the action force (hill) = “action at a distance”
- The wave transports energy and momentum between the hill and site of dissipation via form stresses $\int p' \frac{d\eta\prime}{dx}\, dx$
- The force is given by the decay of the form stress: $\mathrm{F} = \frac{d}{dz} \int p' \frac{d\eta\prime}{dx}\, dx$

Wave energy budget

bar = time mean (mean)
prime = time varying (wave)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z} = b$$

$$\frac{Db}{Dt} = \kappa\nabla^2 b$$

$$\nabla\cdot\mathbf{u} = 0.$$

Let $u = \bar{u} + u'$ where $\bar{u} = \frac{1}{T}\int_0^T u\,dt \Rightarrow \frac{1}{T}\int_0^T u'\,dt = 0 \Rightarrow \frac{1}{T}\int_0^T \bar{u}u'\,dt = 0$

$$\left(\frac{\partial}{\partial t} + (\underline{\bar{u}} + \underline{u'})\cdot\nabla\right)(\bar{u}+u') - f(\bar{v}+v') = -\frac{1}{\rho_0}\frac{\partial}{\partial x}(\bar{p}+p') + \nu\nabla^2(\bar{u}+u')$$

Wave energy budget

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Wave energy budget

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$$\frac{\partial\bar{u}}{\partial t} + \underline{\bar{u}}\cdot\nabla\bar{u} + \underbrace{\overline{\underline{u}'\cdot\nabla u'}}_{\text{wave term}} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial x} + \nu\nabla^2\bar{u}$$

Momentum equation for the “mean” flow

Wave energy budget

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

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$$-$$

$$\frac{\partial\bar{u}}{\partial t} + \underline{\bar{u}}\cdot\nabla\bar{u} + \underbrace{\overline{\underline{u}'\cdot\nabla u'}}_{\text{wave term.}} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial x} + \nu\nabla^2\bar{u}$$

Momentum equation for the “mean” flow

$$=$$

$$\frac{\partial u'}{\partial t} + \underline{u}'\cdot\nabla\bar{u} + \underline{u}\cdot\nabla u' - \overline{\underline{u}'\cdot\nabla u'} - fv' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu\nabla^2 u'$$

Momentum equation for the “wave” flow

Wave energy budget

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

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$$-$$

$$\frac{\partial\bar{u}}{\partial t} + \underline{\bar{u}}\cdot\nabla\bar{u} + \underbrace{\overline{\underline{u}'\cdot\nabla u'}}_{\text{wave term.}} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial x} + \nu\nabla^2\bar{u}$$

Momentum equation for the “mean” flow

$$=$$

$$\left(\frac{\partial u'}{\partial t} + \underline{u}'\cdot\nabla\bar{u} + \underline{u}\cdot\nabla u' - \overline{\underline{u}'\cdot\nabla u'} - fv' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu\nabla^2 u'\right) \times u'$$

$$\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)\frac{u'^2}{2} + u'\underline{u}'\cdot\nabla\bar{u} - fv'u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}u' + \nu u'\nabla^2 u' + \overline{\underline{u}'\cdot\nabla u'}\,u'$$

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$$+\left(\frac{\partial}{\partial t}+\underline{u}\cdot\nabla\right)\frac{v'^2}{2} + v'\underline{u}'\cdot\nabla\bar{v} + fv'u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}v' + \nu v'\nabla^2 v' + \cancel{\overline{\underline{u}'\cdot\nabla v'}\,v'}$$

$$\Rightarrow \frac{D}{Dt}K + \underline{u}'_h(\underline{u}'\cdot\nabla)\underline{\bar{v}} = -\frac{1}{\rho_0}\underline{u}'_h\cdot\nabla p' + \nu\nabla^2 K - \nu|\nabla\underline{u}'_h|^2$$

$O(u'^3)$
Zero time
mean

Wave energy budget

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

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$$\Rightarrow \quad \frac{D}{Dt}K + \underline{u}'_h(\underline{u}'\cdot\nabla)\underline{\bar{v}} = -\frac{1}{\rho_0}\boxed{\underline{u}'_h\cdot\nabla p'} + \nu\nabla^2 K - \nu|\nabla\underline{u}'_h|^2$$

$$\boxed{\underline{u}'_h\cdot\nabla p'} = \underline{u}'\cdot\nabla p' - w'\frac{\partial p'}{\partial z}$$

$$= \nabla\cdot(\underline{u}'p') - p'\boxed{\nabla\cdot\underline{u}'} - w\boxed{\frac{\partial p'}{\partial z}}$$

$$= \nabla\cdot(\underline{u}'p') - w'b'\rho_0$$

Wave KE

$$K = \frac{1}{2}(u'^2 + v'^2)$$

Wave KE budget

$$\frac{D}{Dt}K + \nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right) = -w'b' - \mathbf{u}'_\mathbf{h}(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}} - \nu|\nabla\mathbf{u}'_\mathbf{h}|^2 + \nu\nabla^2 K$$

Wave energy budget

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

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$$\Rightarrow \frac{\bar{D}}{Dt}K + \underline{u}_h'(\underline{u}'\cdot\nabla)\underline{\bar{v}} = -\frac{1}{\rho_0}\boxed{\underline{u}_h'\circ\nabla p'} + \nu\nabla^2 K - \nu|\nabla\underline{u}_h'|^2$$

$$\boxed{\underline{u}_h'\circ\nabla p'} = \underline{u}'\circ\nabla p' - w'\frac{\partial p'}{\partial z}$$

$$= \nabla\circ(\underline{u}'p') - p'\boxed{\nabla\circ\underline{u}'} - w\boxed{\frac{\partial p'}{\partial z}}$$

$$= \nabla\circ(\underline{u}'p') - w'b'\rho_0$$

<u>Wave KE</u>

$$K = \frac{1}{2}(u'^2 + v'^2)$$

<u>Wave KE budget</u>

$$\frac{D}{Dt}K + \underbrace{\nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right)}_{\text{Wave energy flux}} = \underbrace{w'b'}_{\text{Wave buoyancy flux}} - \underbrace{\mathbf{u_h'}(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}}}_{\text{Mean-to-wave exchange}} - \underbrace{\nu|\nabla\mathbf{u_h'}|^2}_{\text{Viscous dissipation}} + \underbrace{\nu\nabla^2 K}_{\text{Diffusion of KE (boundary fluxes)}}$$

Wave energy budget

$$\frac{D}{Dt}K + \nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right) = -w'b' - \mathbf{u}_\mathbf{h}'(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}} - \nu|\nabla\mathbf{u}_\mathbf{h}'|^2 + \nu\nabla^2 K$$

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

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$$\nabla\cdot\mathbf{u} = 0.$$

- Derivation of the wave KE budget did not rely on any assumptions about the flow.
- The PE budget is nastier…

$$\left(\frac{\partial}{\partial t} + (\underline{\bar{u}}+\underline{u}')\cdot\nabla\right)(\bar{b}+b') = K\nabla^2(\bar{b}+b')$$

$$\Rightarrow \frac{\partial\bar{b}}{\partial t} + \underline{\bar{u}}\cdot\nabla\bar{b} + \overline{\underline{u}'\cdot\nabla b'} = K\nabla^2\bar{b}$$

$$\Rightarrow \left(\frac{\partial b'}{\partial t} + \underline{u}\cdot\nabla b' + \underline{u}'\cdot\nabla\bar{b} - \overline{\underline{u}'\cdot\nabla b'} = K\nabla^2 b'\right)\times\frac{b'}{\bar{b}_z}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{b'^2}{2\bar{b}_z}\right) + \frac{\underline{\bar{u}}\cdot\nabla\left(\frac{b'^2}{2}\right)}{\bar{b}_z} + \frac{\underline{u}'b'\cdot\nabla\bar{b}}{\bar{b}_z} = \frac{K}{\bar{b}_z}\left(\nabla^2\left(\frac{b'}{2}\right) - |\nabla b'|^2\right) + \cancel{\frac{\overline{u'\cdot\nabla b'}\,b'}{\bar{b}_z}}$$

$$\frac{\underline{u}'b'}{\bar{b}_z}\cdot\nabla\bar{b} = \frac{\underline{u}_h'b'}{\bar{b}_z}\cdot\nabla_h\bar{b} + \frac{w'b'}{\bar{b}_z}\bar{b}_z$$

Wave energy budget

$$\frac{D}{Dt}K+\nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right)=-w'b'-\mathbf{u}_\mathbf{h}'(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}}-\nu|\nabla\mathbf{u}_\mathbf{h}'|^2+\nu\nabla^2K$$

$$\frac{Du}{Dt}-fv=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\nabla^2u$$

$$\frac{Dv}{Dt}+fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\nu\nabla^2v$$

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$$\frac{Db}{Dt}=\kappa\nabla^2b$$

$$\nabla\cdot\mathbf{u}=0.$$

- Derivation of the wave KE budget did not rely on any assumptions about the flow…
- The PE budget is nastier…

$$\left(\frac{\partial}{\partial t}+(\underline{\bar{u}}+\underline{u}')\cdot\nabla\right)(\bar{b}+b')=\kappa\nabla^2(\bar{b}+b')$$

$$\Rightarrow\ \frac{\partial\bar{b}}{\partial t}+\underline{\bar{u}}\cdot\nabla\bar{b}+\overline{\underline{u}'\cdot\nabla b'}=\kappa\nabla^2\bar{b}$$

$$\Rightarrow\ \left(\frac{\partial b'}{\partial t}+\underline{u}\cdot\nabla b'+\underline{u}'\cdot\nabla\bar{b}-\overline{\underline{u}'\cdot\nabla b'}=\kappa\nabla^2b'\right)\times\frac{b'}{\bar{b}_z}$$

$$\Rightarrow\ \frac{\partial}{\partial t}\left(\frac{b'^2}{2\bar{b}_z}\right)+\frac{\underline{\bar{u}}\cdot\nabla\left(\frac{b'^2}{2}\right)}{\bar{b}_z}+\frac{\underline{u}'b'\cdot\nabla\bar{b}}{\bar{b}_z}=\frac{\kappa}{\bar{b}_z}\left(\nabla^2\left(\frac{b'}{2}\right)-|\nabla b'|^2\right)+\cancel{\frac{\overline{\underline{u}'\cdot\nabla b'\,b'}}{\bar{b}_z}}$$

$$\frac{\underline{u}'b'}{\bar{b}_z}\cdot\nabla\bar{b}=\frac{\underline{u}_h'b'}{\bar{b}_z}\cdot\nabla_h\bar{b}+\frac{w'b'}{\bar{b}_z}\bar{b}_z$$

$\bar{b}_z\simeq N^2=\text{const}.$ Then $P=\frac{b'^2}{2N^2}$

$$\frac{D}{Dt}P=-w'b'+\frac{\underline{u}_h'b'}{N^2}\cdot\nabla_h\bar{b}+\kappa\nabla^2P-\frac{\kappa}{N^2}|\nabla b'|^2$$

Wave energy budget

$$\frac{D}{Dt}K+\nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right)=-w'b'-\mathbf{u}_\mathbf{h}'(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}}-\nu|\nabla\mathbf{u}_\mathbf{h}'|^2+\nu\nabla^2K$$

$$\frac{Du}{Dt}-fv=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\nabla^2u$$

$$\frac{Dv}{Dt}+fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\nu\nabla^2v$$

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z}=b$$

$$\frac{Db}{Dt}=\kappa\nabla^2b$$

$$\nabla\cdot\mathbf{u}=0.$$

- Derivation of the wave KE budget did not rely on any assumptions about the flow…
- The PE budget is nastier…
- N^2 must vary slowly or not at all for this PE budget to be valid (usually okay for internal waves)

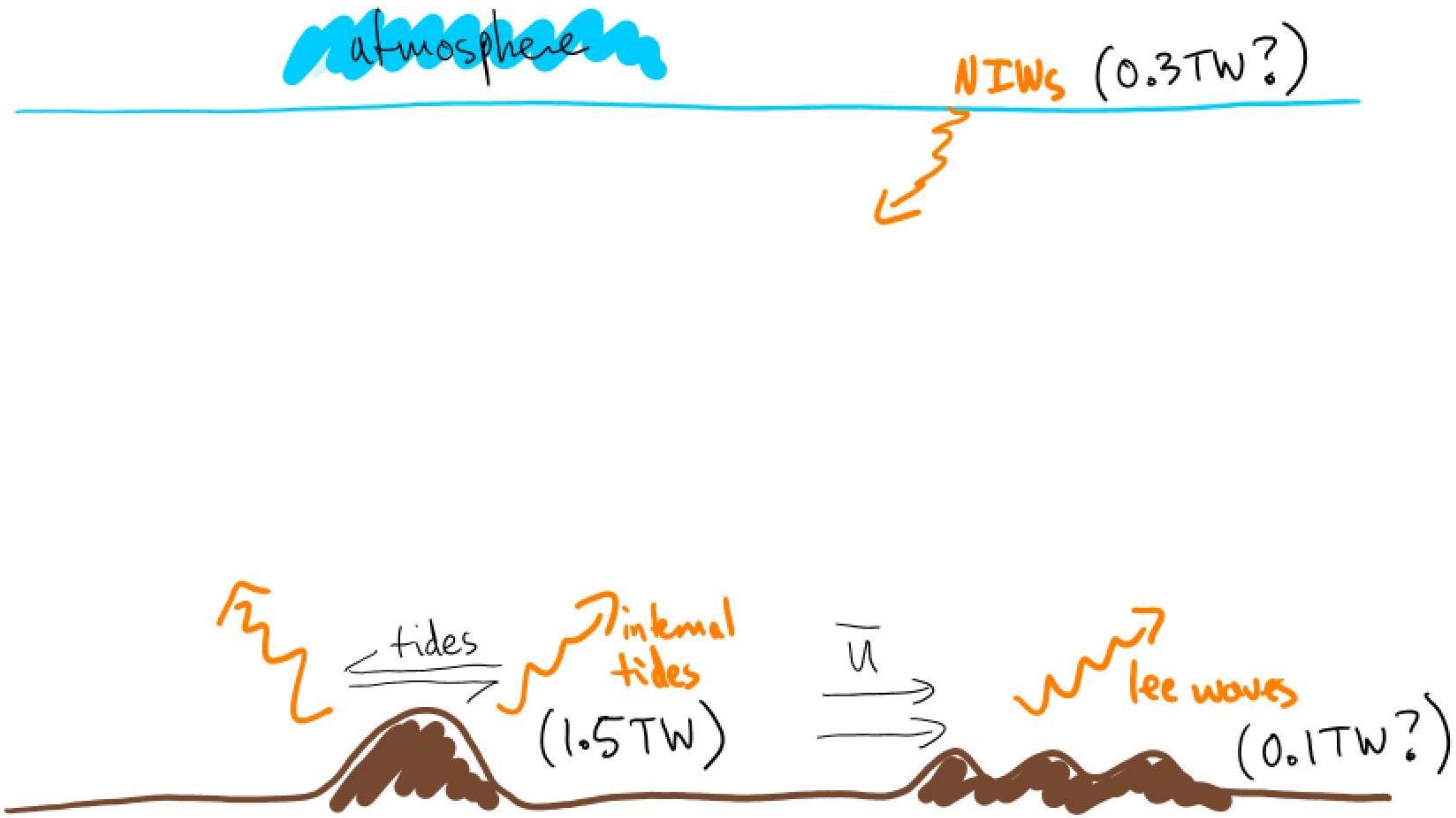
$$\frac{D}{Dt}P=\underbrace{-w'b'}_{\text{Wave buoyancy flux}}\underbrace{-\mathbf{u}_\mathbf{h}'b'\frac{\nabla_h\bar{b}}{N^2}}_{\text{Mean-to-wave exchange}}\underbrace{-\frac{\kappa}{N^2}|\nabla b'|^2}_{\text{Irreversible mixing}}+\underbrace{\kappa\nabla^2P}_{\text{Diffusion of PE (boundary fluxes)}}$$

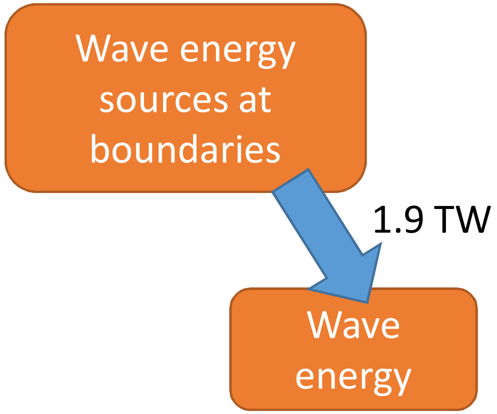
- More general and more exact PE formulations exist; they use a more sophisticated definition of the background state
  - E.g. Hughes, Hogg and Griffiths, 2009.

Ocean internal wave energy budget

$$\frac{\bar{D}}{Dt}K + \nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right) = -w'b' - \mathbf{u}_\mathbf{h}'(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}} - \nu|\nabla\mathbf{u}_\mathbf{h}'|^2 + \nu\nabla^2 K$$

$$\frac{D}{Dt}P = -w'b' - \mathbf{u}_\mathbf{h}'b'\frac{\nabla_h\bar{b}}{N^2} - \frac{\kappa}{N^2}|\nabla b'|^2 + \kappa\nabla^2 P$$




Wave energy sources at boundaries

1.9 TW

Wave energy

Ocean internal wave energy budget

$$\frac{\bar{D}}{Dt}K + \nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right) = -w'b' - \mathbf{u}_\mathbf{h}'(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}} - \nu|\nabla\mathbf{u}_\mathbf{h}'|^2 + \nu\nabla^2 K$$

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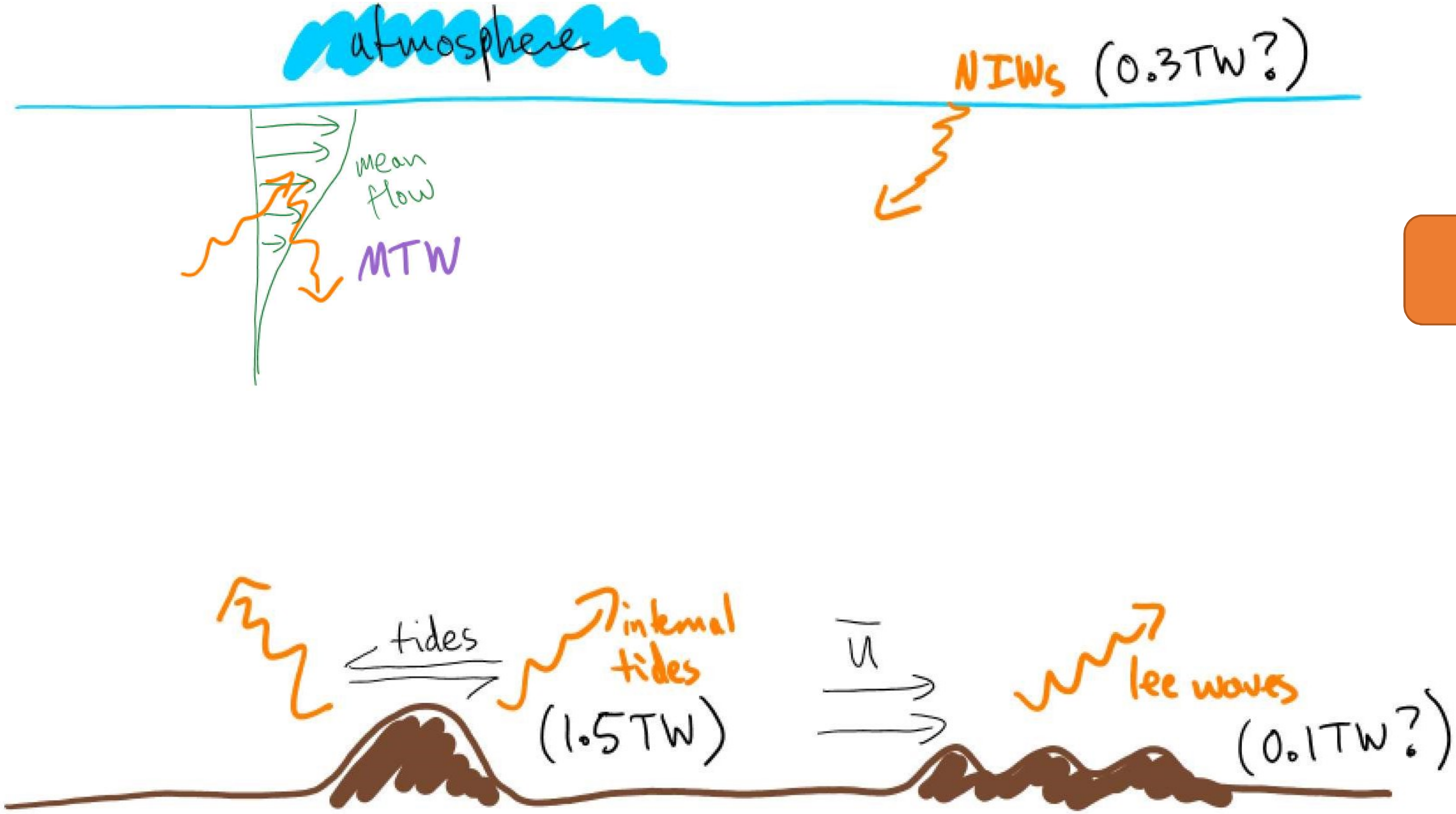

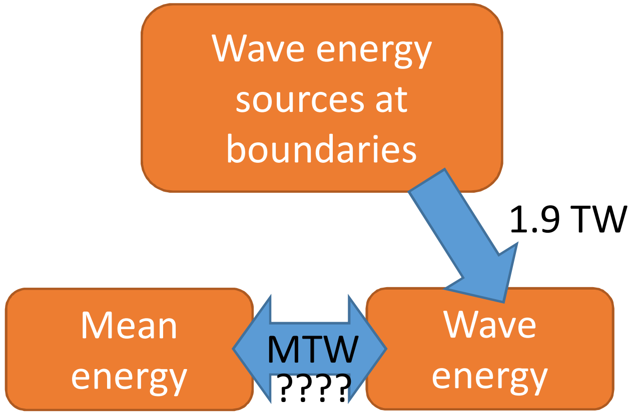

Wave energy sources at boundaries

1.9 TW

Mean energy

MTW ????

Wave energy

Ocean internal wave energy budget

$$\frac{\bar{D}}{Dt}K+\nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right)=-w'b'-\mathbf{u}_{\mathbf{h}}'(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}}-\nu|\nabla\mathbf{u}_{\mathbf{h}}'|^2+\nu\nabla^2K$$

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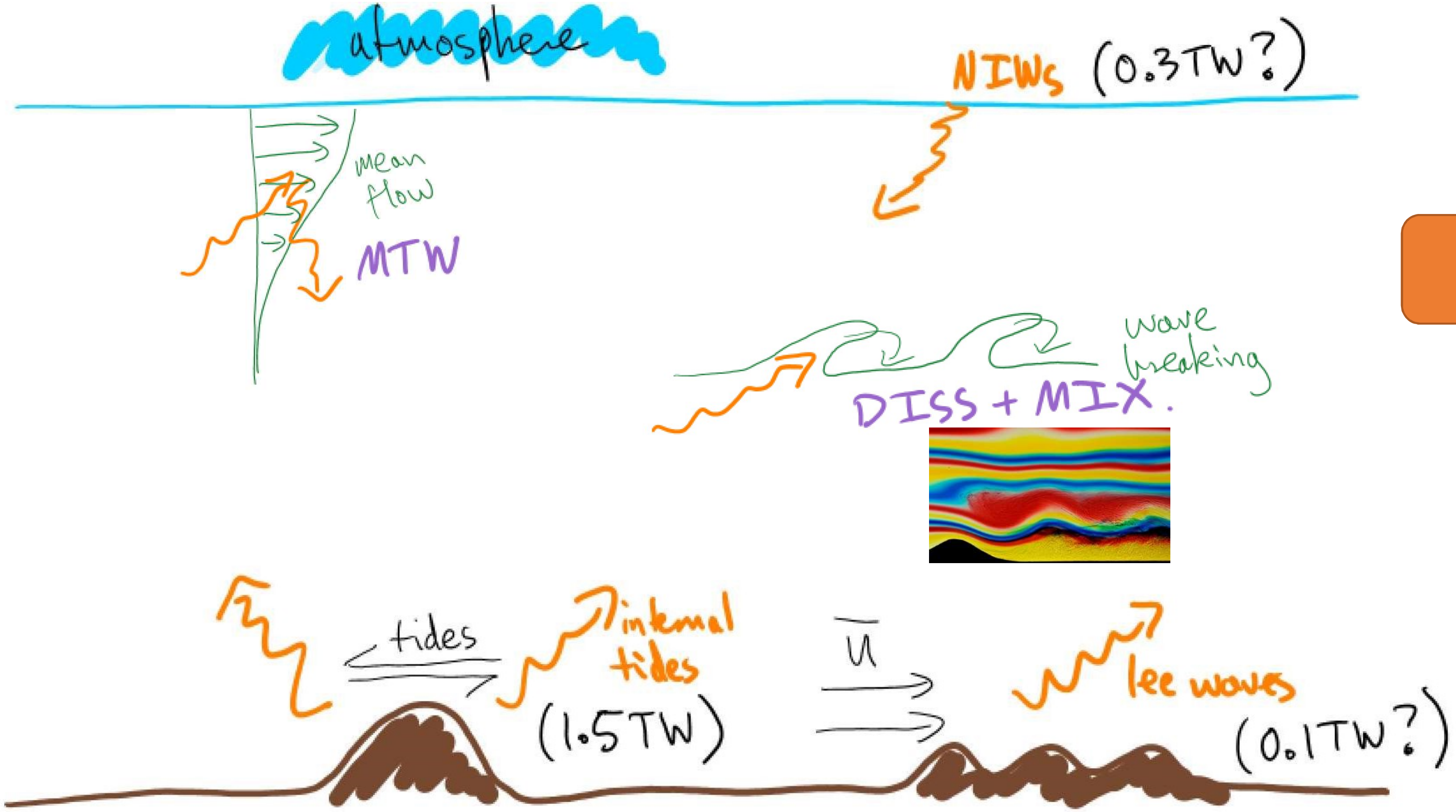

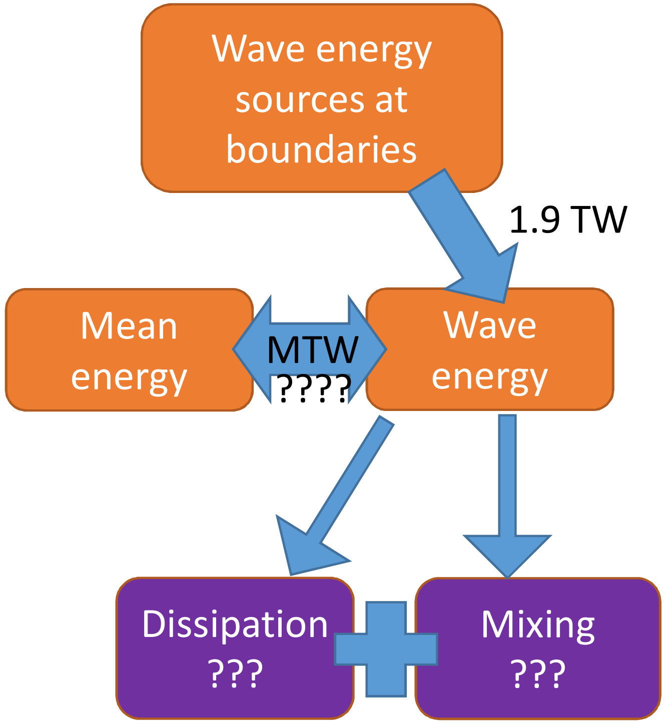

Wave energy sources at boundaries

1.9 TW

Mean energy

MTW ????

Wave energy

Dissipation ???

Mixing ???

Ocean internal wave energy budget

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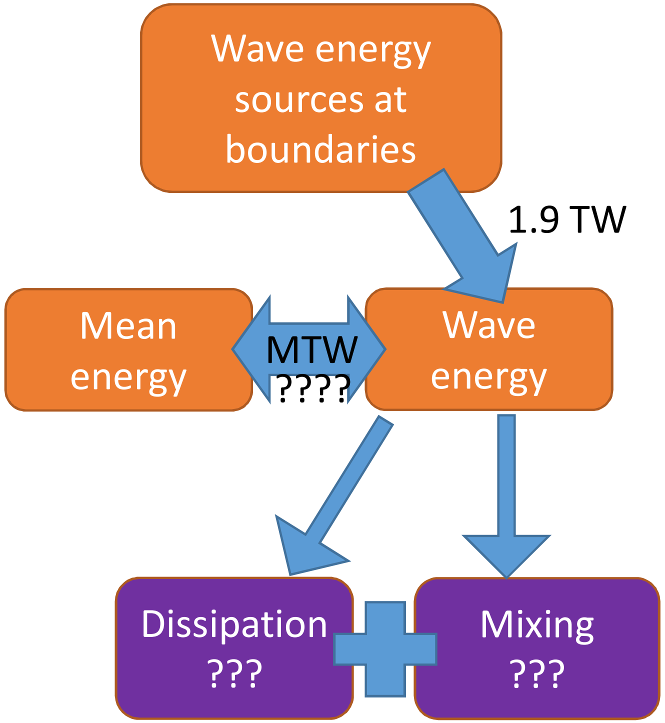
$$\frac{D}{Dt}P=-w'b'-\mathbf{u}_\mathbf{h}'b'\frac{\nabla_h\bar{b}}{N^2}-\frac{\kappa}{N^2}|\nabla b'|^2+\kappa\nabla^2P$$

- Internal waves are able to do mixing in the ocean interior via wave breaking
- Arguably the only process that can do this at scale
- Therefore: crucial to maintaining the ocean's deep overturning (see Annie's lecture later)
- It is the fraction of internal wave energy that goes into mixing which we care about



Wave energy sources at boundaries

1.9 TW

Mean energy

MTW ????

Wave energy

Dissipation ???

Mixing ???

Ocean internal wave energy budget

$$\frac{\bar{D}}{Dt}K+\nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right)=-w'b'-\mathbf{u}_\mathbf{h}'(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}}-\nu|\nabla\mathbf{u}_\mathbf{h}'|^2+\nu\nabla^2K$$

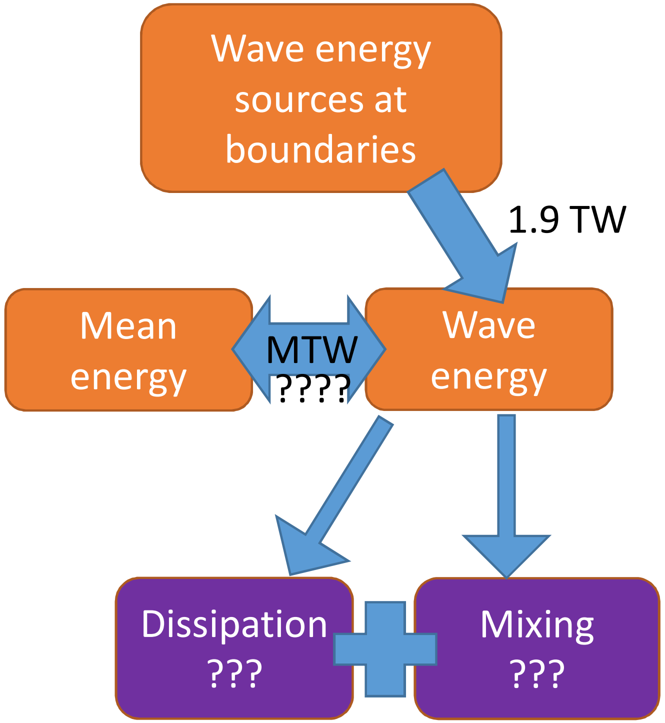
$$\frac{D}{Dt}P=-w'b'-\mathbf{u}_\mathbf{h}'b'\frac{\nabla_h\bar{b}}{N^2}-\frac{\kappa}{N^2}|\nabla b'|^2+\kappa\nabla^2P$$

- The **viscous dissipation (friction)** is unimportant:

$$\epsilon = \rho_0\nu|\nabla u|^2 \sim 10^{-7}W/m^3$$

- This is converted to heating

$$\rho_0\ c_p\ \frac{dT}{dt}=\epsilon\rightarrow\frac{dT}{dt}=\frac{\epsilon}{\rho_0\ c_p}$$



Wave energy sources at boundaries

1.9 TW

Mean energy

MTW ????

Wave energy

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Mixing ???

Ocean internal wave energy budget

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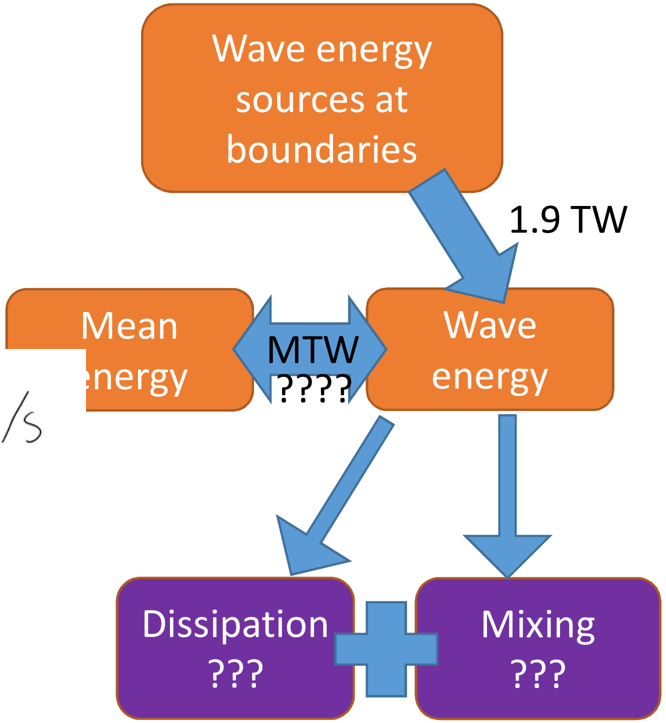
$$\epsilon = \rho_0\nu|\nabla u|^2 \sim 10^{-7}W/m^3$$

- This is converted to heating

$$\rho_0\, c_p\, \frac{dT}{dt} = \epsilon \rightarrow \frac{dT}{dt} = \frac{\epsilon}{\rho_0\, c_p}$$

$$= \frac{10^{-7}}{10^3 \cdot 4000} \sim 2.5\times10^{-14}\ K/s$$

$$= 1\ K \text{ per } 10^6 \text{ years!}$$



Wave energy sources at boundaries

1.9 TW

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Dissipation ???

Mixing ???

Ocean internal wave energy budget

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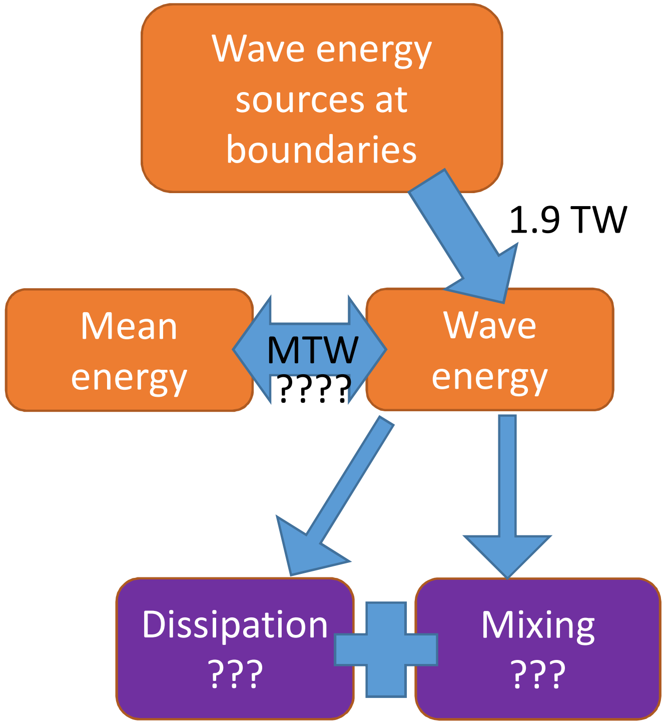
$$\rho_0\,c_p\,\frac{dT}{dt}=\epsilon\rightarrow\frac{dT}{dt}=\frac{\epsilon}{\rho_0\,c_p}$$

- Heats the ocean @ 1K per million years!
- Suppose the same amount of energy goes to mixing
- The irreversible mixing will lift dense water

$$w\,\Delta\rho g=w\,N^2H\,\rho_0=\phi_i\sim10^{-7}W/m^3$$

$$w=\frac{\phi_i}{N^2H\rho_0}=\frac{10^{-7}}{10^{-5}\cdot4\times10^3\cdot10^3}=2.5\times10^{-9}\ m/s$$



Wave energy sources at boundaries

1.9 TW

Mean energy

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Dissipation ???

Mixing ???

Ocean internal wave energy budget

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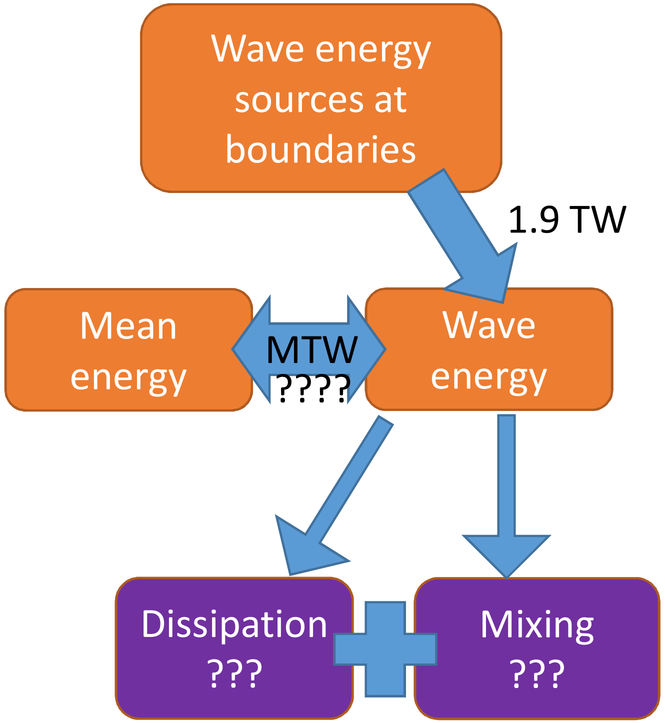
$$w=\frac{\phi_i}{N^2H\rho_0}=\frac{10^{-7}}{10^{-5}\cdot4\times10^3\cdot10^3}=2.5\times10^{-9}\ m/s$$

Area of ocean = $3.5\times10^{14}m^2$ → $w\,A\sim10^6\frac{m^3}{s}\sim1\,Sv$



Wave energy sources at boundaries

1.9 TW

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Dissipation ???

Mixing ???

Ocean internal wave energy budget

$$\frac{\bar{D}}{Dt}K+\nabla\cdot\left(\frac{\mathbf{u}'p'}{\rho_0}\right)=-w'b'-\mathbf{u}_\mathbf{h}'(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}}-\nu|\nabla\mathbf{u}_\mathbf{h}'|^2+\nu\nabla^2K$$

$$\frac{D}{Dt}P=-w'b'-\mathbf{u}_\mathbf{h}'b'\frac{\nabla_h\bar{b}}{N^2}-\frac{\kappa}{N^2}|\nabla b'|^2+\kappa\nabla^2P$$

<u>Summary</u>: the same amount of energy can

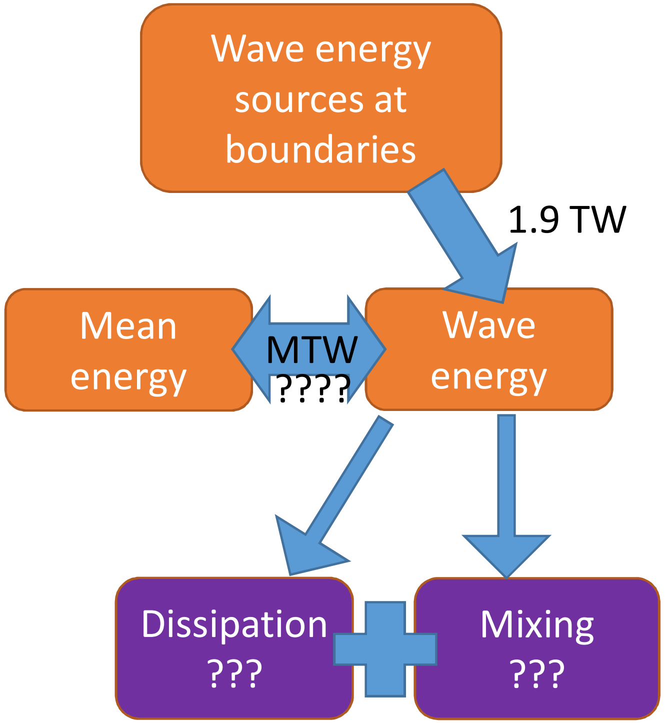
a) Heat the ocean at 1K per millennia (friction)

b) Lift 1 million m^3 per second from the bottom to the top of the ocean (mixing)

We only care about the **mixing**!!

$$\text{mixing efficiency} = \Gamma = \frac{\phi_i}{\epsilon+\phi_i} \sim \frac{\phi_i}{\epsilon}$$

"It is a truth universally acknowledged, that a single wave in possession of a good amount of energy, must be in want of mixing at an efficiency of $\mathbf{\Gamma = 0.2}$"



Note: "Observations" of mixing are usually observations (or models) of dissipation, converted using this assumption.

Distribution of ocean mixing

Internal-Wave-Driven Mixing: Global Geography and Budgets

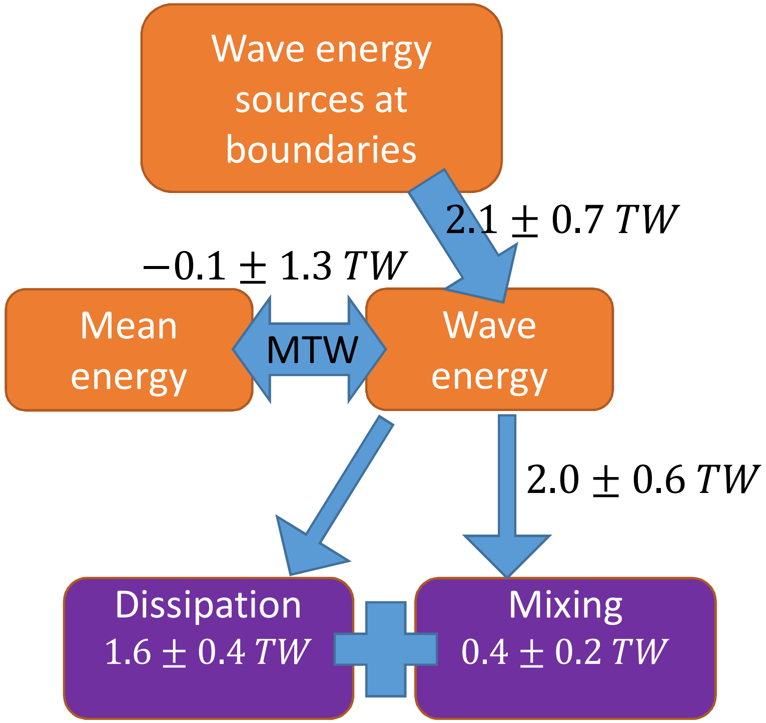
ERIC KUNZE

NorthWest Research Associates, Redmond, Washington

(Manuscript received 13 June 2016, in final form 6 January 2017)

ABSTRACT

Internal-wave-driven dissipation rates ε and diapycnal diffusivities K are inferred globally using a finescale parameterization based on vertical strain applied to ~30 000 hydrographic casts. Global dissipations are 2.0 ± 0.6 TW, consistent with internal wave power sources of 2.1 ± 0.7 TW from tides and wind. Vertically in-



We really don’t know very much….

Unfortunately, the ocean (and models thereof) are very sensitive to the magnitude and distribution of mixing: e.g. Melet et al., 2013

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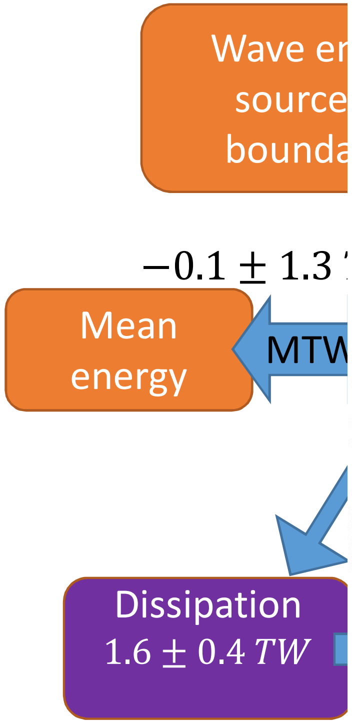
$$\epsilon = \epsilon_0 \frac{\overline{N^2}}{N_0^2} \frac{\langle \xi_z^2 \rangle^2}{\langle \xi_{zGM}^2 \rangle^2} h(R_\omega) L(f, N),$$

“Finescale” dissipation model
The vertical strain is measured at ~10m resolution



Wave en
source
bounda

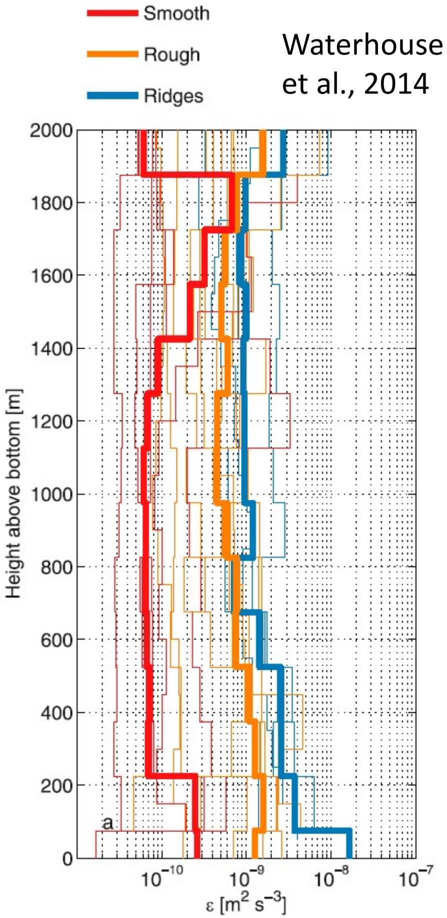
-0.1 ± 1.3

Mean energy

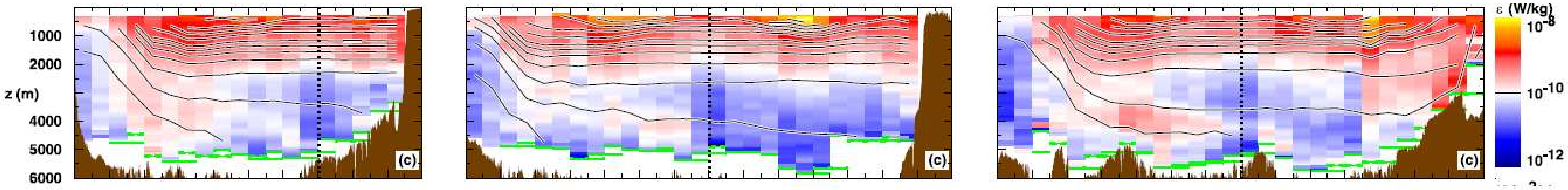
MTW

Dissipation
$1.6 \pm 0.4\, TW$

We really don’t



Waterhouse et al., 2014



Kunze, 2017

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Internal wave energy budget

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