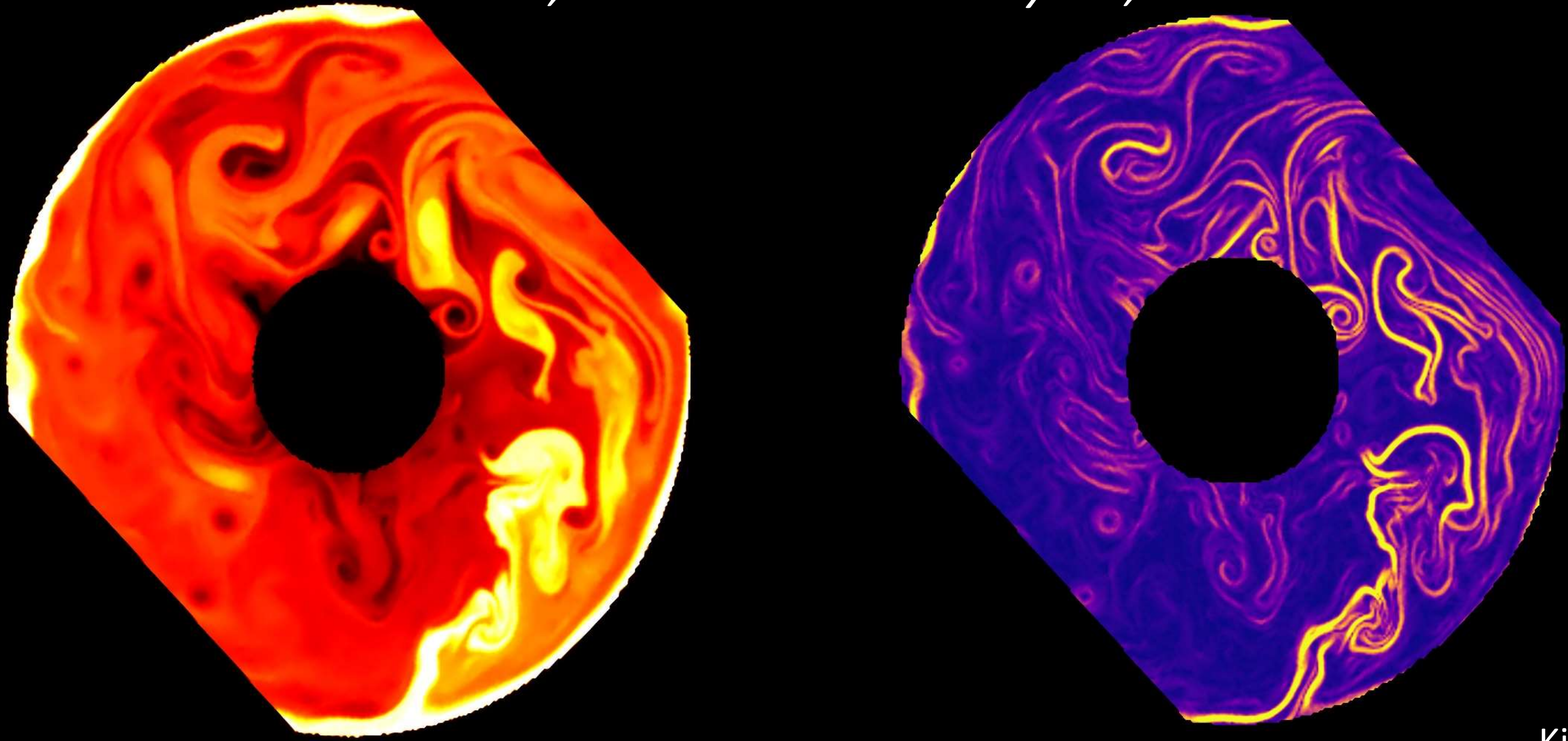
Lecture 10:
Wave-driven circulation

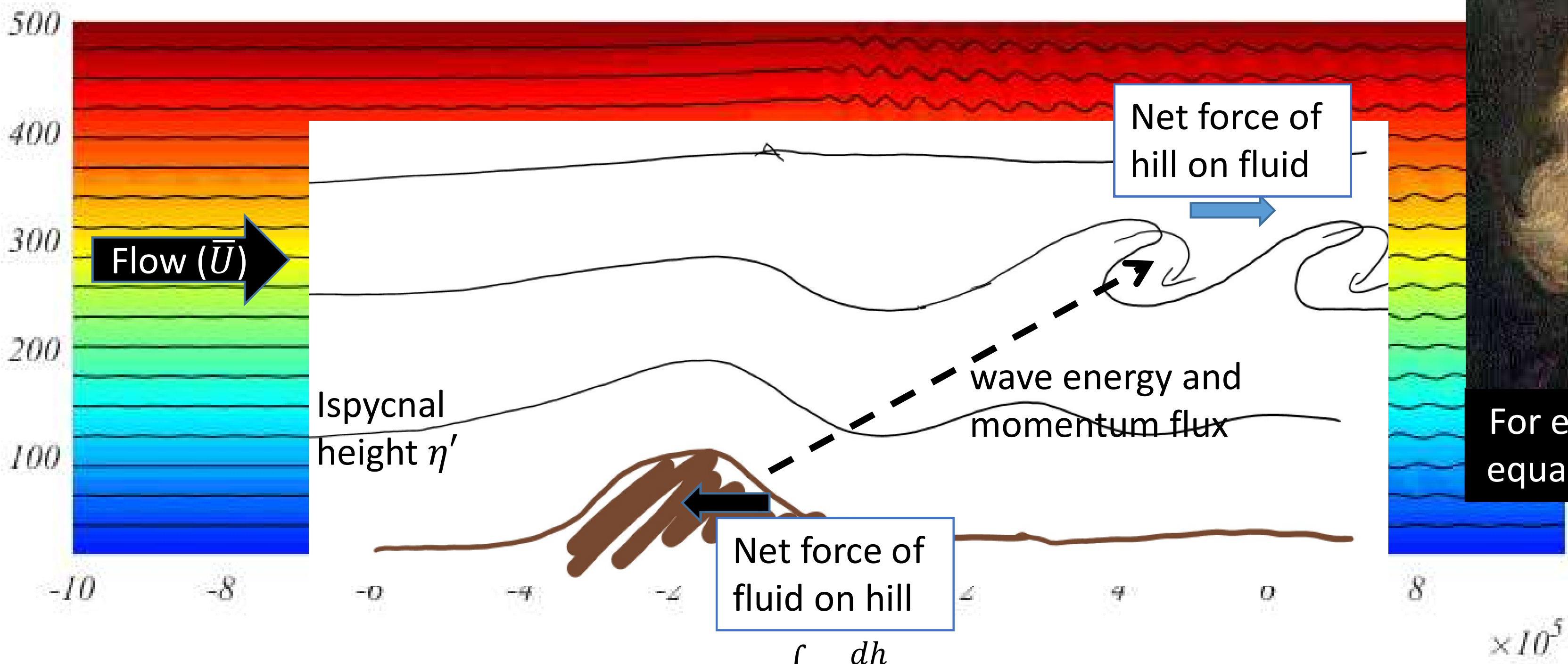
Callum J. Shakespeare

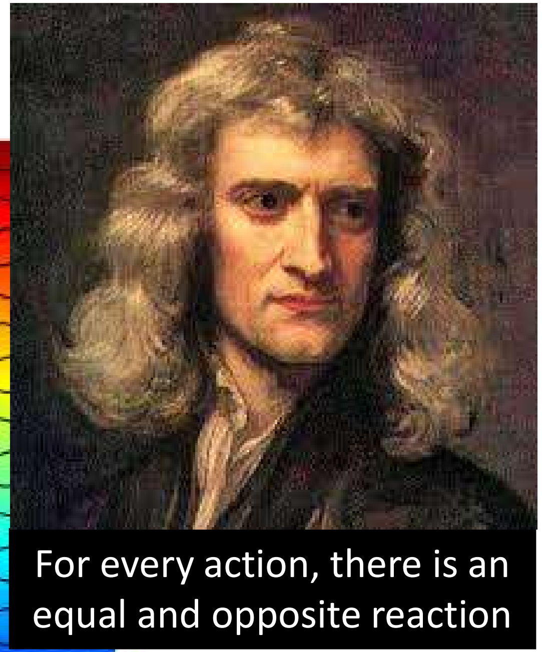
Fellow, Climate and Fluid Physics, ANU



Kial Stewart, GFD Lab

Lee waves as an example



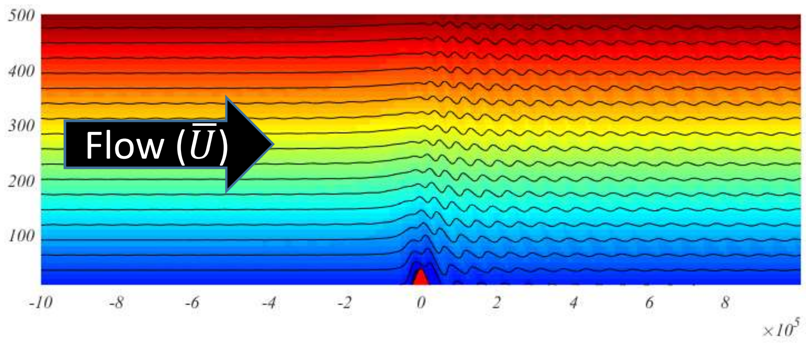


$$\tau = \int p' \frac{dh}{dx}\, dx$$

- A NET reaction force is only felt in the layer where the wave decays/attenuates
- This could be a LONG way from the action force (hill) = "action at a distance"
- The wave transports energy and momentum between the hill and site of dissipation via form stresses $\int p' \frac{d\eta'}{dx}\, dx$
- The force is given by the decay of the form stress: $\mathrm{F} = \frac{d}{dz} \int p' \frac{d\eta'}{dx}\, dx$

Internal wave solutions

- Linearise z-coordinate equations about mean flow $\overline{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity



$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}$$

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)v + fu = \nu\frac{\partial^2 v}{\partial x^2}$$

$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)b + N^2 w = \nu\frac{\partial^2 b}{\partial x^2}$$

$$D = \left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x^2}\right)$$

$$D\,u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

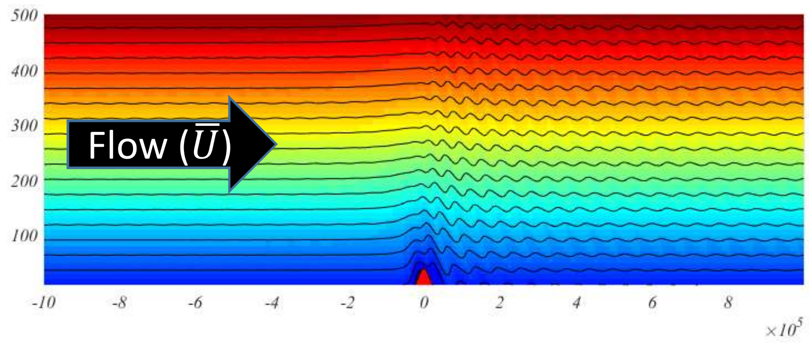
$$Dv + fu = 0$$

$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$Db + N^2 w = 0$$

Internal wave solutions



- Linearise z-coordinate equations about mean flow $\overline{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity

$$D\,u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv + fu = 0$$
$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
$$Db + N^2 w = 0$$

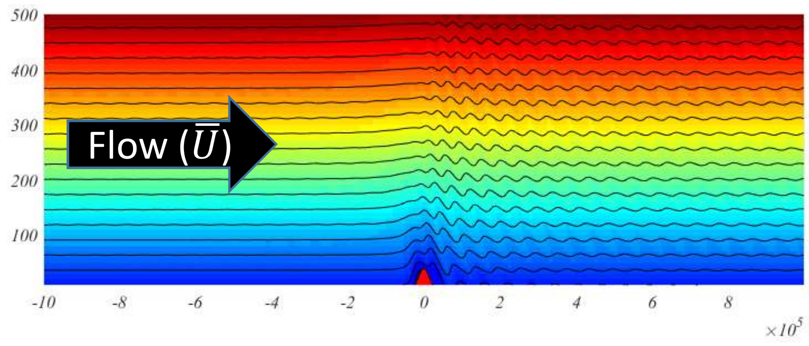
$$D^2 u - fDv = -D\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv = -f\,u$$

$$D^2 u + f^2 u = -D\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$(D^2 + f^2)\frac{\partial u}{\partial z} = -D\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x \partial z}$$

$$D = \left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x^2}\right)$$

Internal wave solutions



- Linearise z-coordinate equations about mean flow $\bar{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity

$$D\,u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv + fu = 0$$
$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
$$Db + N^2 w = 0$$

$$D^2 u - fDv = -D\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv = -f\,u$$

$$D^2 u + f^2 u = -D\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

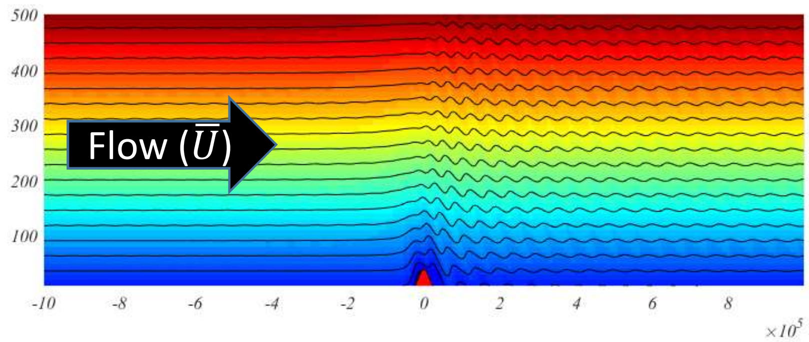
$$(D^2 + f^2)\frac{\partial u}{\partial z} = -D\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x \partial z}$$

$$(D^2 + f^2)\frac{\partial u}{\partial z} = -D\frac{\partial b}{\partial x}$$

$$D = \left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x^2}\right)$$

Internal wave solutions

- Linearise z-coordinate equations about mean flow $\overline{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity



$$D\,u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv + fu = 0$$
$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
$$Db + N^2 w = 0$$

$$D^2 u - fDv = -D\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv = -f\,u$$

$$D^2 u + f^2 u = -D\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$(D^2 + f^2)\frac{\partial u}{\partial z} = -D\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x \partial z}$$

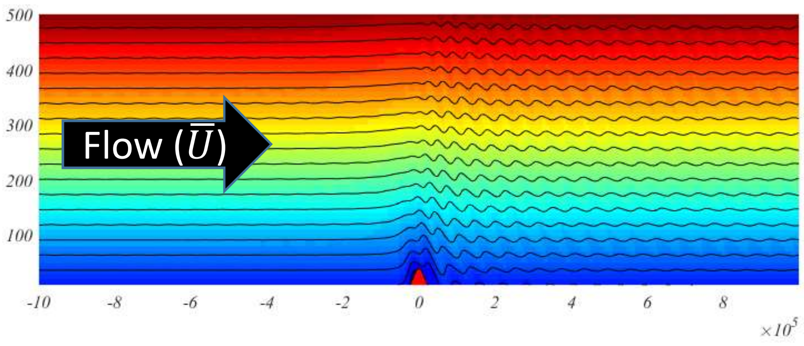
$$(D^2 + f^2)\frac{\partial u}{\partial z} = -D\frac{\partial b}{\partial x}$$

$$(D^2 + f^2)\frac{\partial}{\partial z}\frac{\partial u}{\partial x} = N^2\frac{\partial^2 w}{\partial x}$$

$$D = \left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x^2}\right)$$

Internal wave solutions

- Linearise z-coordinate equations about mean flow $\overline{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity



$$D\,u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv + fu = 0$$
$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
$$Db + N^2 w = 0$$

$$D^2 u - fDv = -D\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv = -f\,u$$

$$D^2 u + f^2 u = -D\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$(D^2 + f^2)\frac{\partial u}{\partial z} = -D\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x \partial z}$$

$$(D^2 + f^2)\frac{\partial u}{\partial z} = -D\frac{\partial b}{\partial x}$$

$$(D^2 + f^2)\frac{\partial}{\partial z}\frac{\partial u}{\partial x} = N^2\frac{\partial^2 w}{\partial x}$$

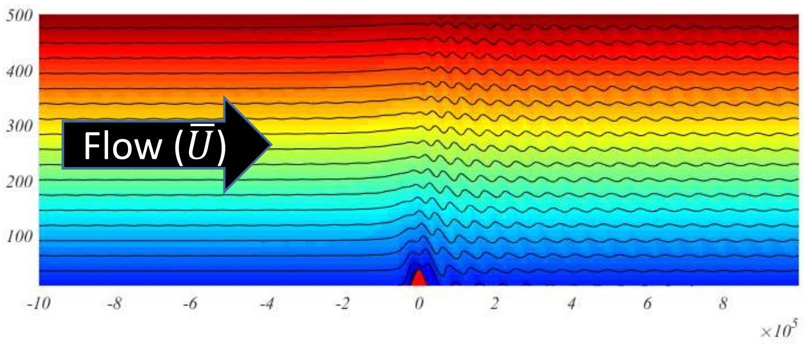
$$(D^2 + f^2)\frac{\partial^2 w}{\partial z^2} + N^2\frac{\partial^2 w}{\partial x} = 0$$

Internal waves in a continously stratified flow

$$D = \left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x^2}\right)$$

Dispersion relation

- Linearise z-coordinate equations about mean flow $\overline{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity



$$(D^2 + f^2)\frac{\partial^2 w}{\partial z^2} + N^2 \frac{\partial^2 w}{\partial x} = 0 \qquad D = \left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x^2}\right)$$

$$\text{Let} \quad w = \widehat{w}\, e^{i(kx+\quad z-\omega t)}$$

$$-m^2\left((-i\,\omega + i\,k\overline{U} + \nu k^2)^2 + f^2\right) - k^2 N^2 = 0$$

$$\omega = k\overline{U} \pm \sqrt{f^2 + \frac{k^2N^2}{m^2}} \boxed{-\,i\,\nu k^2}$$

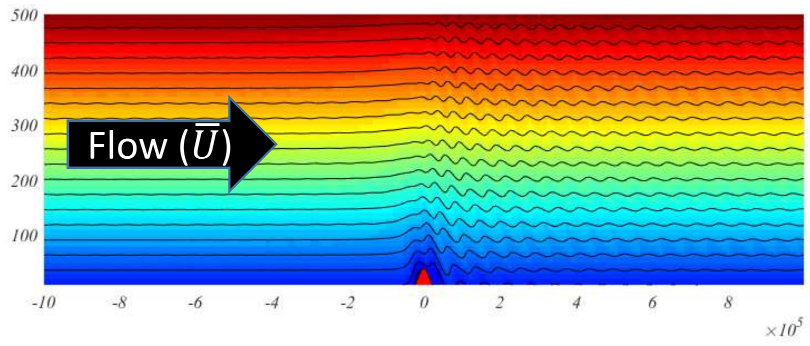
What is the effect of this?

Recall: Internal wave dispersion relation for 2-layered model

$$\omega = k\overline{U} \pm \sqrt{f^2 + k^2 g' h_1}$$

Dispersion relation

- Linearise z-coordinate equations about mean flow $\overline{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity



$$(D^2 + f^2)\frac{\partial^2 w}{\partial z^2} + N^2 \frac{\partial^2 w}{\partial x} = 0 \qquad D = \left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x} - \nu \frac{\partial^2}{\partial x^2}\right)$$

$$\text{Let} \quad w = \widehat{w}\, e^{i(kx + m\,z - \omega t)}$$

$$-m^2\left((-i\,\omega + i\,k\overline{U} - \nu k^2)^2 + f^2\right) - k^2 N^2 = 0$$

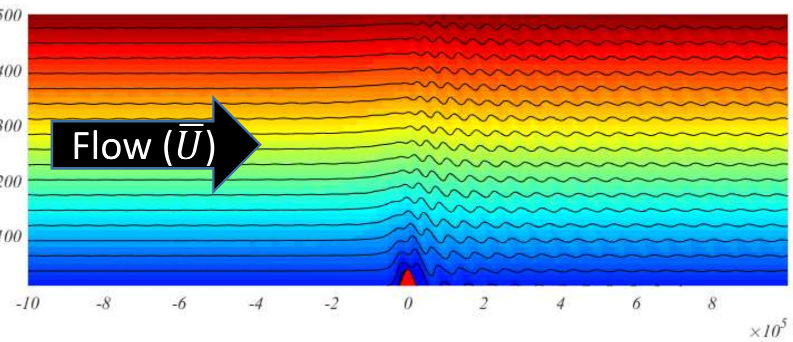
$$\omega = k\overline{U} \pm \sqrt{f^2 + \frac{k^2 N^2}{m^2}} - i\,\nu k^2$$

What is the effect of this? (Wave decays with time)

$$m^2 = \frac{k^2 N^2}{\left((\,\omega - \,k\overline{U} + i\nu k^2)^2 - f^2\right)} = \frac{k^2 N^2}{\left((\,\omega - \,k\overline{U})^2 - f^2 + 2i\nu k^2(\omega - k\,\overline{U})\right)}$$

Dispersion relation

- Linearise z-coordinate equations about mean flow $\bar{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity



$$(D^2 + f^2)\frac{\partial^2 w}{\partial z^2} + N^2 \frac{\partial^2 w}{\partial x^2} = 0 \qquad D = \left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x} - \nu \frac{\partial^2}{\partial x^2}\right)$$

Let $w = \hat{w}\, e^{i(kx + m z - \omega t)}$

$$-m^2\left((-i\,\omega + i\,k\bar{U} - \nu k^2)^2 + f^2\right) - k^2 N^2 = 0$$

$$\omega = k\bar{U} \pm \sqrt{f^2 + \frac{k^2 N^2}{m^2}} \boxed{- i\,\nu k^2}$$

What is the effect of this? (Wave decays with time)

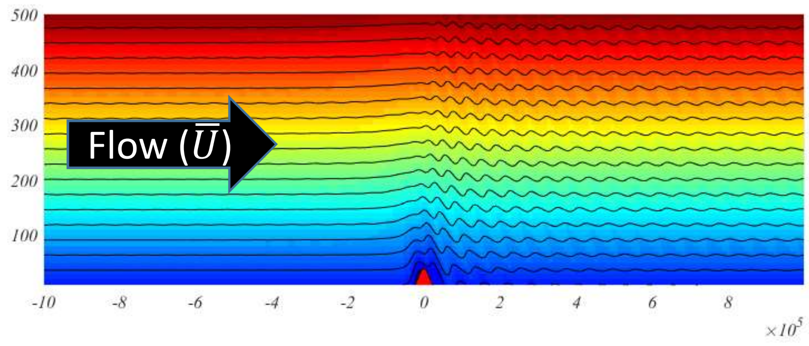
Small dissipation approximation

$$m^2 = \frac{k^2 N^2}{((\omega - k\bar{U} + i\nu k^2)^2 - f^2)} = \frac{k^2 N^2}{((\omega - k\bar{U})^2 - f^2 + 2i\nu k^2(\omega - k\,\bar{U}))} \simeq \frac{k^2 N^2}{(\omega - k\bar{U})^2 - f^2}\left(1 - \frac{2i\nu k^2(\omega - k\,\bar{U})}{(\omega - k\bar{U})^2 - f^2}\right)$$

$$m \simeq \pm \frac{|k| N}{\sqrt{(\omega - k\bar{U})^2 - f^2}}\left(1 - \frac{i\nu k^2(\omega - k\,\bar{U})}{(\omega - k\bar{U})^2 - f^2}\right) = \pm m_0(1 - \gamma i)$$

Wave decays with depth

Internal wave solutions



- Linearise z-coordinate equations about mean flow $\overline{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity

$$(D^2 + f^2)\frac{\partial^2 w}{\partial z^2} + N^2\frac{\partial^2 w}{\partial x} = 0 \qquad \text{Let} \quad w = \widehat{w}\, e^{i(kx+m\,z-\omega t)}$$

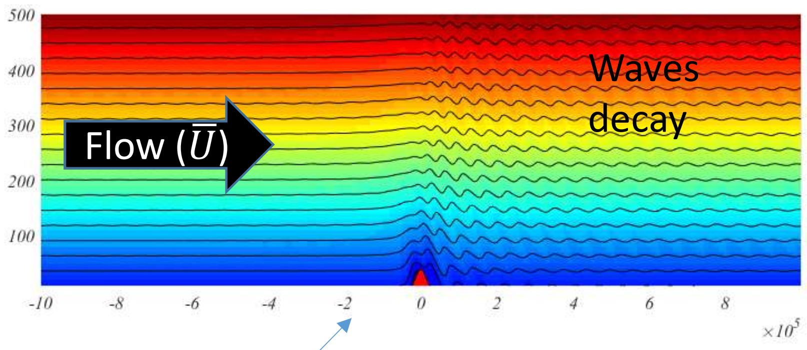
$$m \simeq -\frac{|k|N}{\sqrt{(\omega - k\overline{U})^2 - f^2}}\left(1 - \frac{i\nu k^2(\omega - k\,\overline{U})}{(\omega - k\overline{U})^2 - f^2}\right) = -m_0(1 - \gamma i)$$

$$w = \widehat{w}\, e^{i(kx - m_0 z - \omega t) -}$$

Wave decays as it propagates upwards

Lee wave solutions

- Linearise z-coordinate equations about mean flow $\overline{U}$ and stratification $\frac{\partial b}{\partial z} = N^2$
- Assume no variation in y (ridge)
- Horizontal viscosity only = diffusivity



Solution plotted here

$$(D^2 + f^2)\frac{\partial^2 w}{\partial z^2} + N^2 \frac{\partial^2 w}{\partial x} = 0$$

Let $w = \hat{w}\, e^{i(kx + m\,z - \omega t)}$

$$m \simeq -\frac{|k|N}{\sqrt{(\,\omega - \,k\overline{U})^2 - f^2}}\left(1 - \frac{i\nu k^2(\omega - k\,\overline{U})}{(\,\omega - \,k\overline{U})^2 - f^2}\right) = -m_0(1 - \gamma i)$$

$$w = \hat{w}\, e^{i(kx - \quad {}_0 z - \omega t) -}$$

Wave decays as it propagates upwards

Let $\underline{\omega = 0}$ and we need a boundary condition

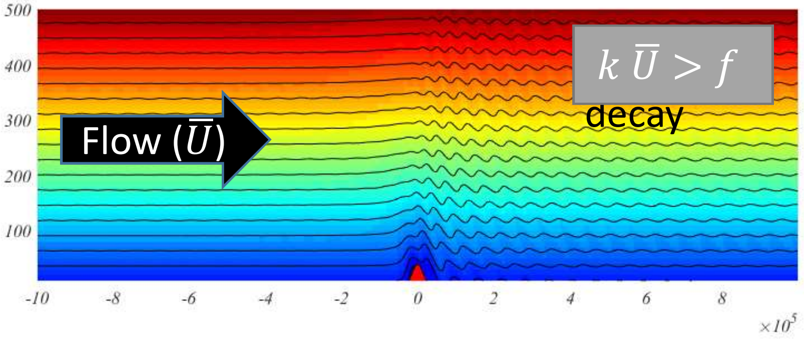
$$w_0 = (u + \overline{U}\,)\frac{dh}{dx} \simeq \overline{U}\frac{dh}{dx} \rightarrow \hat{w} = i\,k\overline{U}\,\hat{h}$$

$$\rightarrow w = i\,k\overline{U}\,\hat{h}\,e^{i(kx - m_0 z) - \gamma z}$$

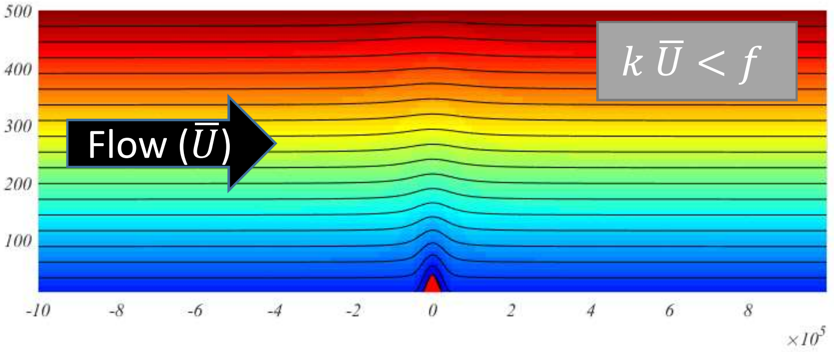
Lee wave solutions

$$m_0 = \frac{|k|N}{\sqrt{(k\overline{U})^2 - f^2}}$$

$$w = i\, k\overline{U}\, \hat{h}\, e^{i(kx - m_0 z) - \gamma z}$$



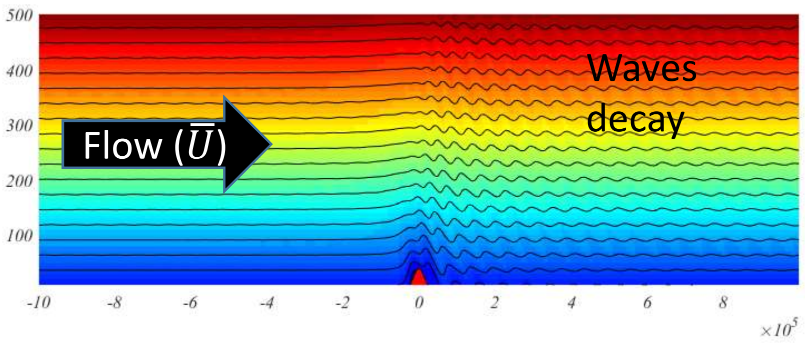
- The wavenumber is only real (waves exist) for $k\,\overline{U} > f$
- This is what we mean by “fast enough”



Lee wave form stress

Solution for vertical velocity: $$w = \int i\, k\overline{U}\, \hat{h}\, e^{i(kx-m_0 z)-\gamma z}\, dk = \int \hat{w}\, dk$$

Form stress: $$F = \int p \frac{\partial \eta'}{\partial x}\, dx$$



Equation tool kit:

$$D\, u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

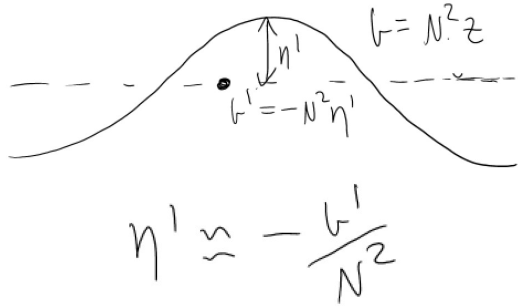
$$Dv + fu = 0$$

$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

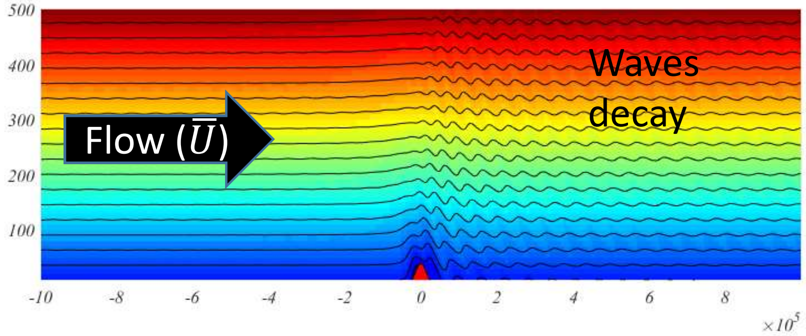
$$Db + N^2 w = 0$$

$$D \simeq \overline{U}\frac{\partial}{\partial x}$$



Lee wave form stress

Solution for vertical velocity: $$w = \int i\, k\overline{U}\, \hat{h}\, e^{i(kx - m_0 z) - \gamma z}\, dk = \int \hat{w}\, dk$$



Form stress: $$F = \int p \frac{\partial \eta'}{\partial x}\, dx$$

Equation tool kit:

$$D\, u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$Dv + fu = 0$$

$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

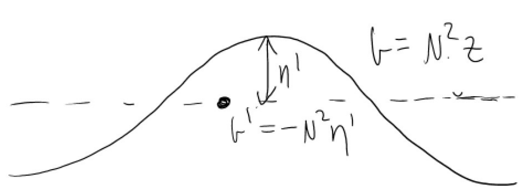
$$Db + N^2 w = 0$$

$$D \simeq \overline{U}\frac{\partial}{\partial x}$$

$$F = \int p \frac{\partial \eta'}{\partial x}\, dx = \int p\, \frac{-1}{N^2}\frac{\partial b}{\partial x} dx$$

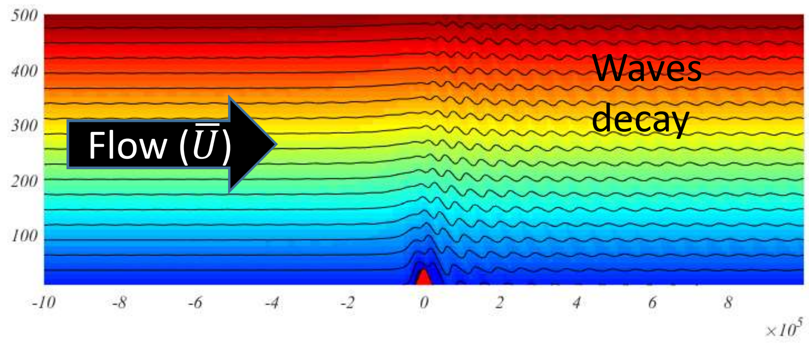
$$= \int p \frac{w}{\overline{U}} dx$$

$$= \frac{1}{\overline{U}} \int wp\, dx$$



$$\eta' \simeq -\frac{b'}{N^2}$$

Lee wave form stress



Solution for vertical velocity: $$w = \int i\, k\bar{U}\, \hat{h}\, e^{i(kx-m_0 z)-\gamma z}\, dk = \int \hat{w}\, dk$$

Form stress: $$F = \int p \frac{\partial \eta'}{\partial x}\, dx$$

Equation tool kit:

$$D\, u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$Dv + fu = 0$$

$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$Db + N^2 w = 0$$

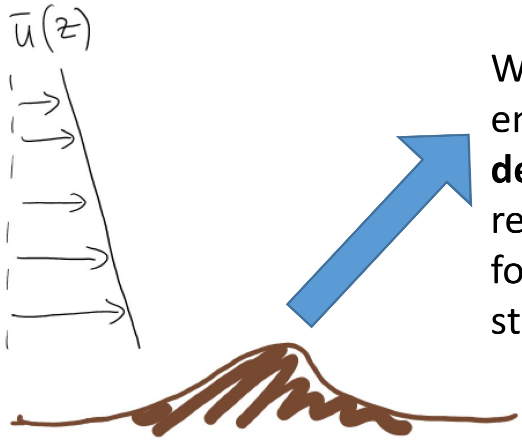
$$D \simeq \bar{U}\frac{\partial}{\partial x}$$

$$F = \int p \frac{\partial \eta'}{\partial x}\, dx = \int p\, \frac{-1}{N^2}\frac{\partial b}{\partial x} dx$$

$$= \int p \frac{w}{\bar{U}} dx$$

$$= \frac{1}{\bar{U}}\int wp\, dx$$

$$\text{Form stress} = \frac{\text{wave energy flux}}{\bar{U}}$$



Wave energy flux **decreases** relative to form stress!

e.g. Bretherton (1969)

Lee wave form stress

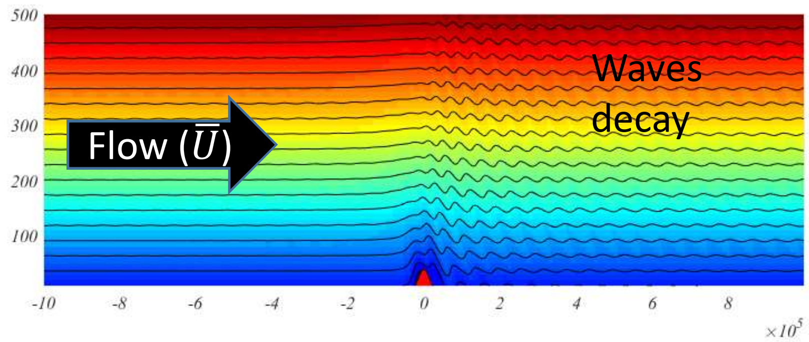
Solution for vertical velocity: $$w = \int i\, k\overline{U}\, \hat{h}\, e^{i(kx - m_0 z) - \gamma z}\, dk = \int \hat{w}\, dk$$

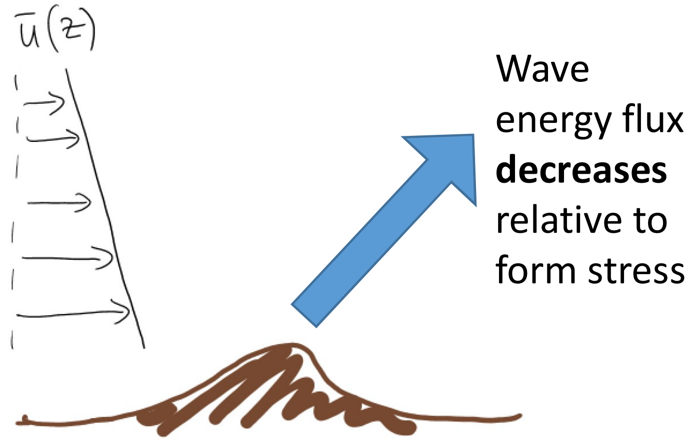
Form stress: $$F = \int p \frac{\partial \eta'}{\partial x}\, dx = \frac{1}{\overline{U}} \int wp\, dx$$

Equation tool kit:

$$\hat{b} = \frac{-i\, m_0}{\rho_0} \hat{p} \qquad \mathrm{i}\, k\overline{U}\hat{b} + N^2 \hat{w} = 0$$

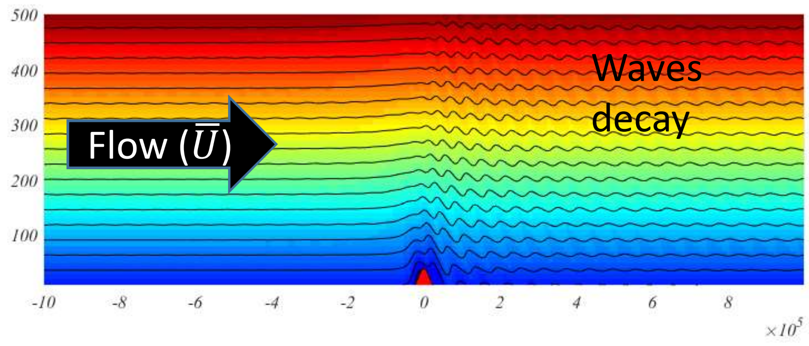
$$D\, u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv + fu = 0$$
$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
$$Db + N^2 w = 0$$
$$D \simeq \overline{U}\frac{\partial}{\partial x}$$





$$\text{Form stress} = \frac{\text{wave energy flux}}{\overline{U}}$$

Lee wave form stress



Solution for vertical velocity: $$w = \int i\, k\overline{U}\, \hat{h}\, e^{i(kx - m_0 z) - \gamma z}\, dk = \int \hat{w}\, dk$$

Form stress: $$F = \int p \frac{\partial \eta'}{\partial x}\, dx = \frac{1}{\overline{U}} \int wp\, dx = \frac{1}{2\pi\overline{U}} \int \hat{w}\, \hat{p}^*\, dk$$

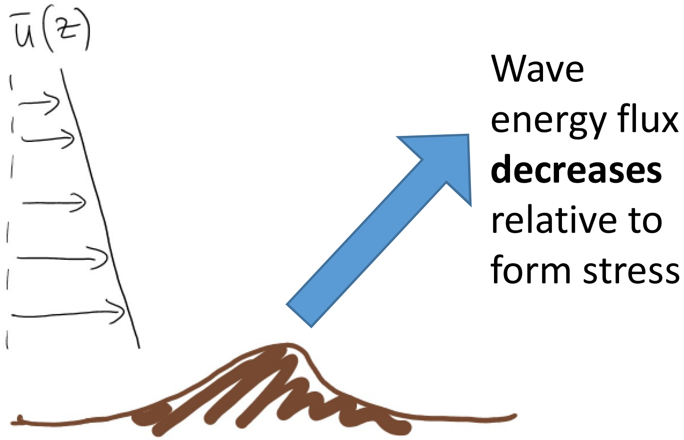
Equation tool kit:

$$D\, u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv + fu = 0$$
$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
$$Db + N^2 w = 0$$
$$D \simeq \overline{U}\frac{\partial}{\partial x}$$

$$\hat{b} = \frac{-i\, m_0}{\rho_0}\hat{p} \qquad \mathrm{i}\, k\overline{U}\hat{b} + N^2\hat{w} = 0$$

$$\hat{p} = -\rho_0 \frac{N^2}{m_0 k\overline{U}}\hat{w}$$

$$\hat{w}\, \hat{p}^* = -\rho_0 \frac{N^2}{m_0 k\overline{U}}|\hat{w}|^2$$

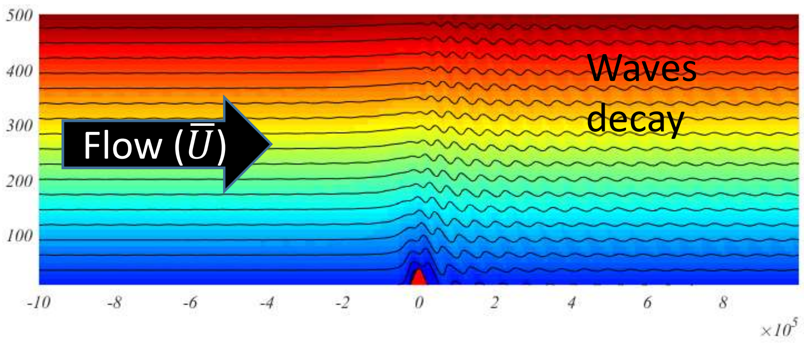


$$\text{Form stress} = \frac{\text{wave energy flux}}{\overline{U}}$$

Lee wave form stress

Solution for vertical velocity: $w = \int i\, k\overline{U}\, \hat{h}\, e^{i(kx - m_0 z) - \gamma z}\, dk = \int \hat{w}\, dk$

Form stress: $F = \int p \frac{\partial \eta'}{\partial x}\, dx = \frac{1}{\overline{U}} \int wp\, dx = \frac{1}{2\pi\overline{U}} \int \hat{w}\, \hat{p}^*\, dk$



Equation tool kit:

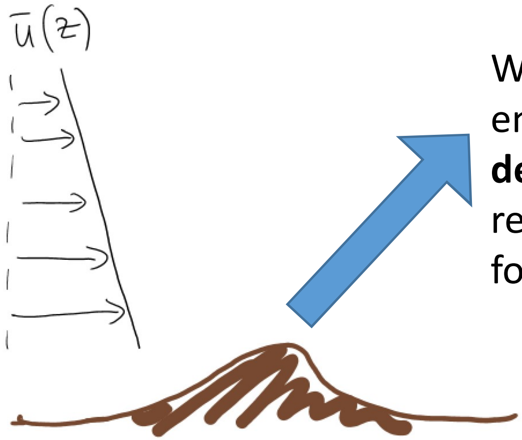
$$D\,u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$
$$Dv + fu = 0$$
$$b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
$$Db + N^2 w = 0$$
$$D \simeq \overline{U}\frac{\partial}{\partial x}$$

$$\hat{b} = \frac{-i\, m_0}{\rho_0}\hat{p} \qquad \mathrm{i}\, k\overline{U}\hat{b} + N^2\hat{w} = 0$$

$$\hat{p} = -\rho_0 \frac{N^2}{m_0 k\overline{U}}\hat{w}$$

$$\hat{w}\, \hat{p}^* = -\rho_0 \frac{N^2}{m_0 k\overline{U}} |\hat{w}|^2$$

$$\text{Energy flux} = \frac{-\rho_0 N^2}{m_0} \overline{U}|k|\left|\hat{h}\right|^2 e^{-2\gamma z}$$



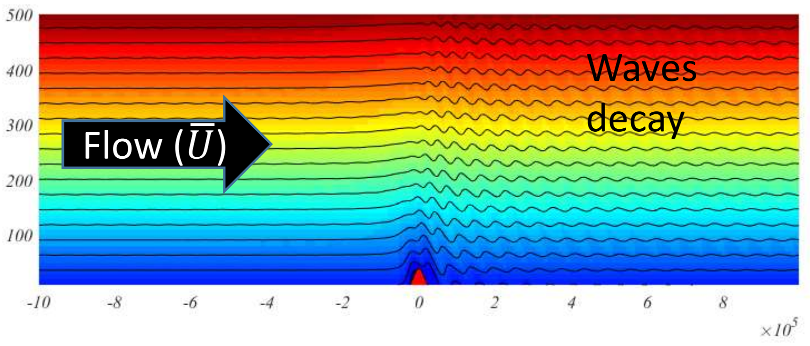
Wave energy flux **decreases** relative to form stress

$$\text{Form stress} = \frac{\text{wave energy flux}}{\overline{U}}$$

Lee wave form stress

$$\text{Energy flux} = \frac{-\rho_0 N^2}{m_0} \bar{U} |k| |\hat{h}|^2 \, e^{-2\gamma z}$$

$$\text{Form stress} = \frac{-\rho_0 N^2}{m_0} |k| |\hat{h}|^2 \, e^{-2\gamma z}$$

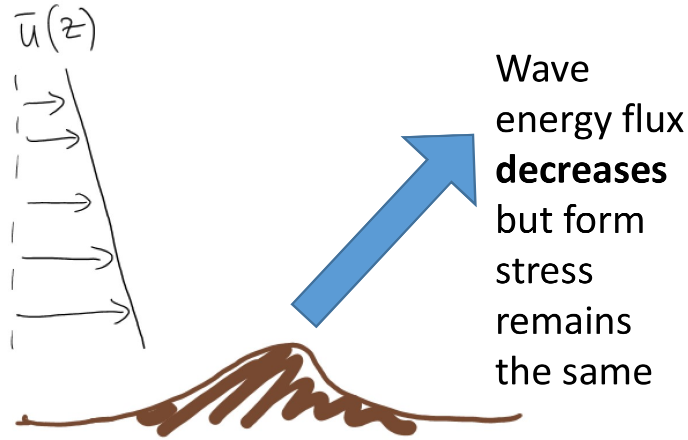


- Energy flux decays due to dissipation γ>0 and due to changes in mean flow (mean-to-wave exchanges)
- Form stress only decays due to dissipation γ>0

Conclusions:

- A wave can lose (or gain) energy to a mean flow without changing its total form stress
  - Waves **do not** possess momentum e.g. McIntyre, 1981: “On the wave momentum myth”

- **Form stress only decays (= force on the mean flow) if there is dissipation!**
  - “non-acceleration theorem” e.g. Andrews and McIntyre (1978)



$$\text{Form stress} = \frac{\text{wave energy flux}}{\bar{U}}$$

The impact of waves (time varying flow) on the mean flow

- Previously we came up with the mean momentum equation:

$$\frac{\partial \bar{u}}{\partial t} + \bar{\underline{u}} \cdot \nabla \bar{u} + \overline{\underline{u}' \cdot \nabla u'} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu \nabla^2 \bar{u}$$

wave term

- Rearranging we have that

$$\frac{\bar{D}}{Dt}\bar{u} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu \nabla^2 \bar{u} - \nabla \cdot \left(\overline{\underline{u}' u'}\right)$$

Reynolds stresses

- In the vertical direction this is $\overline{u'w'}$ = vertical 'momentum flux'
- Are we done? Does this equal the stress from the hill?

The impact of waves (time varying flow) on the mean flow

Does $-p' \frac{d\eta'}{dx} = \rho_0 u'w'$?

Form stress Reynolds stress

The impact of waves (time varying flow) on the mean flow

Does $-p'\frac{d\eta'}{dx} = \rho_0 u'w'$?

Form stress — Reynolds stress

$$-\int p'\frac{\partial \eta'}{\partial x}dx = +\int \frac{\partial p'}{\partial x}\eta' dx$$

Integrate by parts

$$= +\rho_0 \int \frac{Du'}{Dt}\eta' - f v'\eta' \, dx$$

$$\frac{\partial p'}{\partial x} = -\rho_0\left(\frac{Du'}{Dt} - f\,v'\right)$$

$$= +\rho_0 \int \frac{D}{Dt}(u'\eta') - u'\frac{D\eta'}{Dt} - f v'\eta' \, dx$$

$$= +\rho_0 \int \frac{D}{Dt}(u'\eta') - u'w' - f v'\eta' \, dx$$

$$w' = \frac{D\eta'}{Dt}$$

$$\frac{1}{\rho_0} p'\frac{\partial \eta'}{\partial x} = u'w' + f v'\eta' - \frac{D}{Dt}(u'\eta')$$

No!

The impact of waves (time varying flow) on the mean flow

$$\frac{1}{\rho_0} p' \frac{\partial \eta'}{\partial x} = u'w' + f v'\eta' - \frac{D}{Dt}(u'\eta')$$

Stress on an isopycnal

Stress on a z level

Corrections/translations

Can we re-write our previous momentum balance in terms of the isopycnal stress?

$$\frac{\bar{D}}{Dt}\bar{u} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \nabla\cdot\left(\overline{\underline{u}'u'}\right)$$

$$\frac{\bar{D}\bar{u}}{Dt} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \nabla\cdot\left(\overline{\underline{u}'_h u'}\right) - \frac{\partial}{\partial z}\left(\overline{\frac{p'}{\rho_0}\frac{\partial \eta'}{\partial x} - f v'\eta' + \frac{D}{Dt}(u'\eta')}\right)$$

$$\Rightarrow \frac{D}{Dt}\underbrace{\left(\bar{u} + \frac{\partial}{\partial z}\left(\overline{u'\eta'}\right)\right)}_{\bar{u}^*} - f\underbrace{\left(\bar{v} + \frac{\partial}{\partial z}\left(\overline{v'\eta'}\right)\right)}_{\bar{v}^*} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \nabla\cdot\left(\overline{\underline{u}'_h u' + \hat{\underline{k}}\frac{p'}{\rho_0}\frac{\partial \eta'}{\partial x}}\right)$$

The impact of waves (time varying flow) on the mean flow

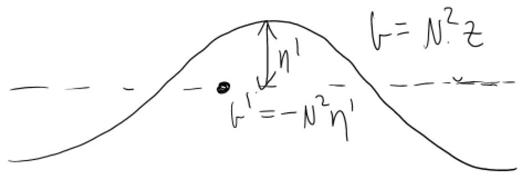
Momentum balance with stress at fixed z

$$\frac{\bar{D}}{Dt}\bar{u} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \nabla\cdot(\overline{\underline{u}'u'})$$

Momentum balance with stress at fixed isopycnal (form stress)

$$\frac{D\bar{u}^*}{Dt} - f\bar{v}^* = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \nabla\cdot\underline{F}$$

$$\underline{F} = \left(\overline{u'u'}, \overline{u'v'}, \frac{1}{\rho_0}\overline{p'\frac{\partial \eta'}{\partial x}}\right) \quad \text{Form stress}$$



$$\eta' \simeq -\frac{b'}{N^2}$$

Residual Flow

$$\bar{u}^* = \bar{u} + \frac{\partial}{\partial z}\overline{u'\eta'}$$

$$\bar{v}^* = \bar{v} + \frac{\partial}{\partial z}\overline{v'\eta'}$$

The impact of waves (time varying flow) on the mean flow

$$\frac{1}{\rho_0} p' \frac{\partial \eta'}{\partial x} = u'w' + f v'\eta' - \frac{D}{Dt}(u'\eta')$$

Momentum balance with stress at fixed z

$$\frac{\bar{D}}{Dt}\bar{u} - f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu \nabla^2 \bar{u} - \nabla \cdot \left(\overline{\underline{u}'u'} \right)$$

Momentum balance with stress at fixed isopycnal (form stress)

$$\frac{D\bar{u}}{Dt} - f\bar{v}^* = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu \nabla^2 \bar{u} - \nabla \cdot \underline{F}_{EP}$$

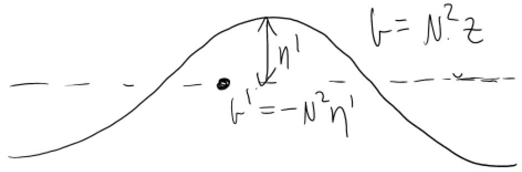
$$\underline{F}_{EP} = \left(\overline{u'u'}, \overline{u'v'}, \overline{u'w' - \frac{v'b'}{N^2}} \right)$$

Eliassen-Palm (EP) Flux

Residual Flow

$$\bar{u}^* = \bar{u} + \frac{\partial}{\partial z}\overline{u'\eta'} = \bar{u} - \frac{\overline{u'b'}}{N^2}$$

$$\bar{v}^* = \bar{v} + \frac{\partial}{\partial z}\overline{v'\eta'} = \bar{v} - \frac{\overline{v'b'}}{N^2}$$



$$\eta' \simeq -\frac{b'}{N^2}$$

The impact of waves (time varying flow) on the mean flow

You get the same result by integrating from the bottom to some isopycnal (as you might do when calculating the MOC in density space….)

$$\int_h^{\eta_1} \frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \nabla^2 u \, d\eta$$

$$\frac{D}{Dt}\int_h^{\eta} u\,dz - f\int_h^{\eta} v\,dz = -\frac{1}{\rho_0}\int_h^{\eta}\frac{\partial p}{\partial x}dz$$

*It must be true that $\frac{D\eta}{Dt} = 0$ for this step

$$= -\frac{1}{\rho_0}\int_h^{\bar{\eta}}\frac{\partial p}{\partial x}dz - \frac{1}{\rho_0}\int_{\bar{\eta}}^{\bar{\eta}+\eta'}\frac{\partial p}{\partial x}dz$$

Split the mean and wave parts

$$= -\frac{1}{\rho_0}\int_h^{\bar{\eta}}\frac{\partial p}{\partial x}dz - \eta'\frac{\partial p}{\partial x}$$

Make the small amplitude approximation

$$\frac{D}{Dt}\left(\frac{\partial}{\partial\bar{\eta}}\int_h^{\eta} u\,dz\right) - f\frac{\partial}{\partial\bar{\eta}}\int_h^{\eta} v\,dz = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - \frac{\partial}{\partial\bar{\eta}}\left(\eta'\frac{\partial p}{\partial x}\right)$$

Take the gradient with respect to mean isopycnal heights

Flow integrated from bottom to height

Form stress

What does this all mean?

Suppose we have a steady mean flow, with small Rossby number (neglect advection)

$$\cancel{\frac{D\bar{u}}{Dt}} - f\bar{v}^* = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \cancel{\nu\nabla^2}\bar{u} - \nabla \cdot \underline{F}_{EP}$$

$$\underline{F}_{EP} = \left(\overline{u'u'},\ \overline{u'v'},\ \overline{u'w' - \frac{v'b'}{N^2}}\right)$$

Take a zonal average (denote by [ ]) and assume the domain is periodic (e.g. the ACC)

$$-f\left[\bar{v}^*\right] = -\frac{\partial}{\partial y}\left[\overline{u'u'}\right] - \frac{\partial}{\partial z}\left[\overline{u'w' - \frac{v'b'}{N^2}}\right] + \nu\frac{\partial^2 \bar{u}}{\partial y^2} + \nu\frac{\partial^2 \bar{u}}{\partial z^2}$$

What does this all mean?

Suppose we have a steady mean flow, with small Rossby number (neglect advection)

$$\cancel{\frac{D\bar{u}}{Dt}} - f\bar{v}^* = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \cancel{\nu\nabla^2\bar{u}} - \nabla\cdot \underline{F}_{EP}$$

$$\underline{F}_{EP} = \left(\overline{u'u'},\ \overline{u'v'},\ \overline{u'w' - \frac{v'b'}{N^2}}\right)$$

Take a zonal average (denote by [ ]) and assume the domain is periodic (e.g. the ACC)

$$-f[\bar{v}^*] = \cancel{-\frac{\partial}{\partial y}\left[\overline{u'u'}\right]} - \frac{\partial}{\partial z}\left[\overline{u'w' - \frac{v'b'}{N^2}}\right] + \cancel{\nu\frac{\partial^2\bar{u}}{\partial y^2}} + \nu\frac{\partial^2\bar{u}}{\partial z^2}$$

$$\bar{v}^* = \frac{\partial\bar{\psi}^*}{\partial z} \qquad \bar{\psi}^* = \frac{1}{f}\left[\overline{u'w' - \frac{v'b'}{N^2}} - \nu\frac{\partial\bar{u}}{\partial z}\right]$$ "Residual" streamfunction

$$\bar{\psi}^*_{surf} = \frac{1}{f}\left[\overline{u'w' - \frac{v'b'}{N^2}} - \frac{\tau}{\rho_0}\right]$$ Value at surface

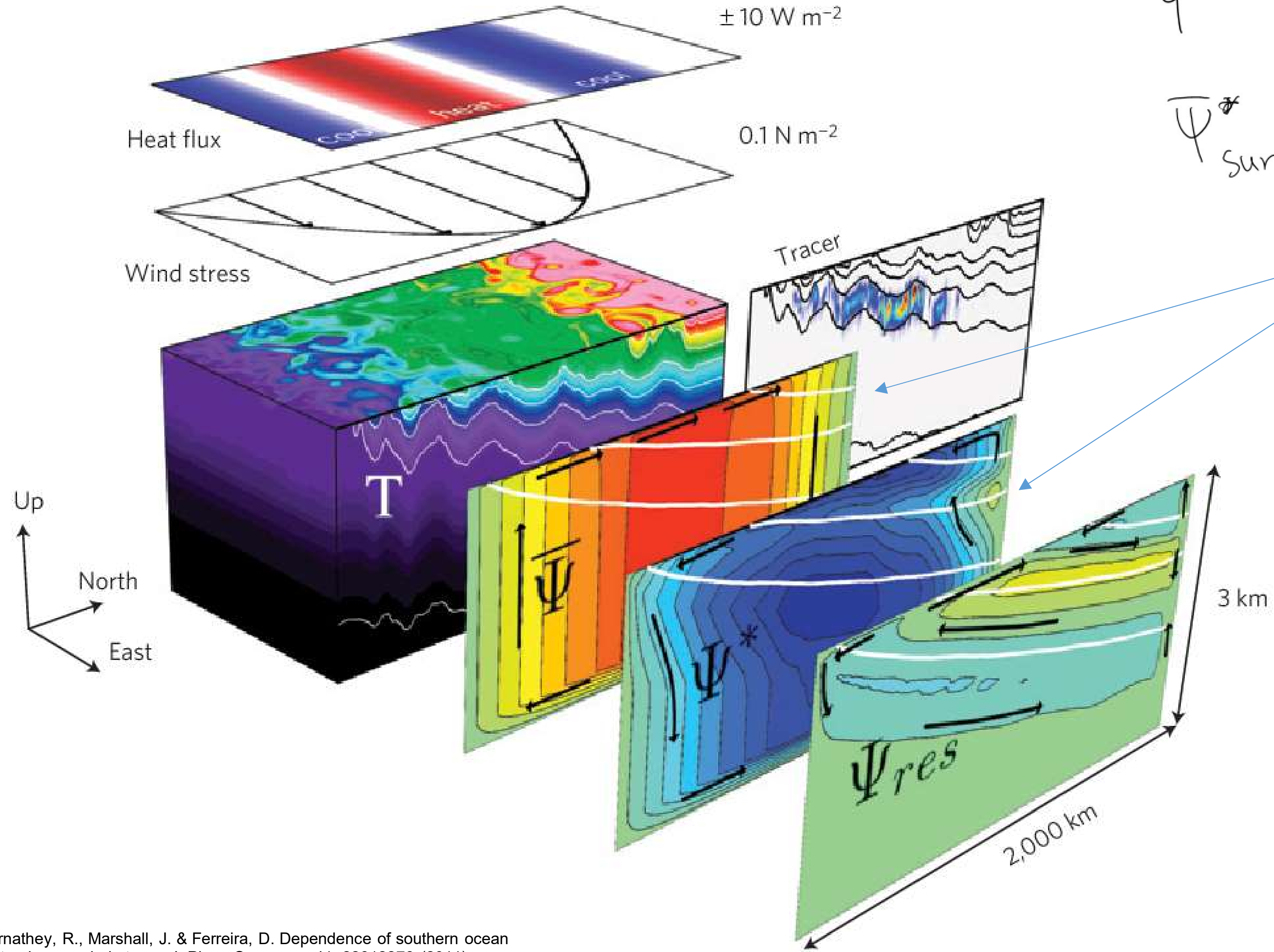
Wave (or eddy) driven circulation

Wind driven circulation

Eddy-driven circulation

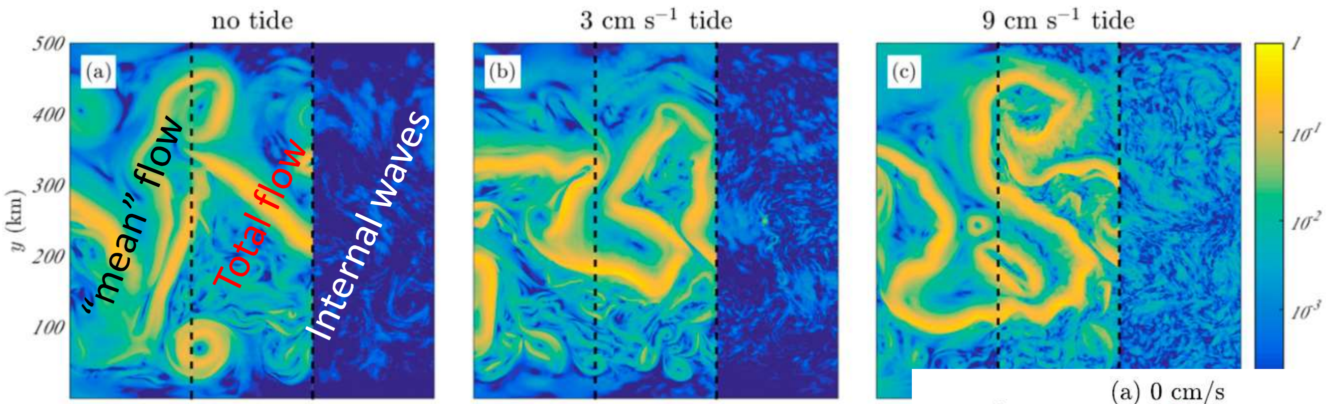
$$\overline{\Psi}^* = \frac{1}{f}\left[\overline{u'w'} - \frac{\overline{v'b'}}{N^2} - \nu\frac{\partial \bar{u}}{\partial z}\right]$$

$$\overline{\Psi}^*_{surf} = \frac{1}{f}\left[\overline{u'w'} - \frac{\overline{v'b'}}{N^2} - \frac{\tau}{\rho_0}\right]$$



Abernathey, R., Marshall, J. & Ferreira, D. Dependence of southern ocean overturning on wind stress. J. Phys. Oceanogr. 41, 22612278 (2011)

Internal-wave driven circulation

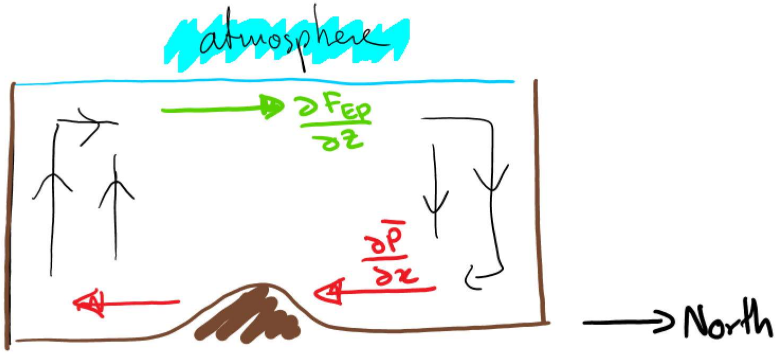


$$\frac{D\bar{u}}{Dt} - f\bar{v}^* = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \nabla\cdot\underline{F}_{EP}$$

$$\underline{F}_{EP} = \left(\overline{u'u'},\ \overline{u'v'},\ \overline{u'w' - \frac{v'b'}{N^2}}\right)$$

x,y mean

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial z}(\bar{u}\bar{w}) - f\bar{v}^* = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial z}F^x_{EP}$$



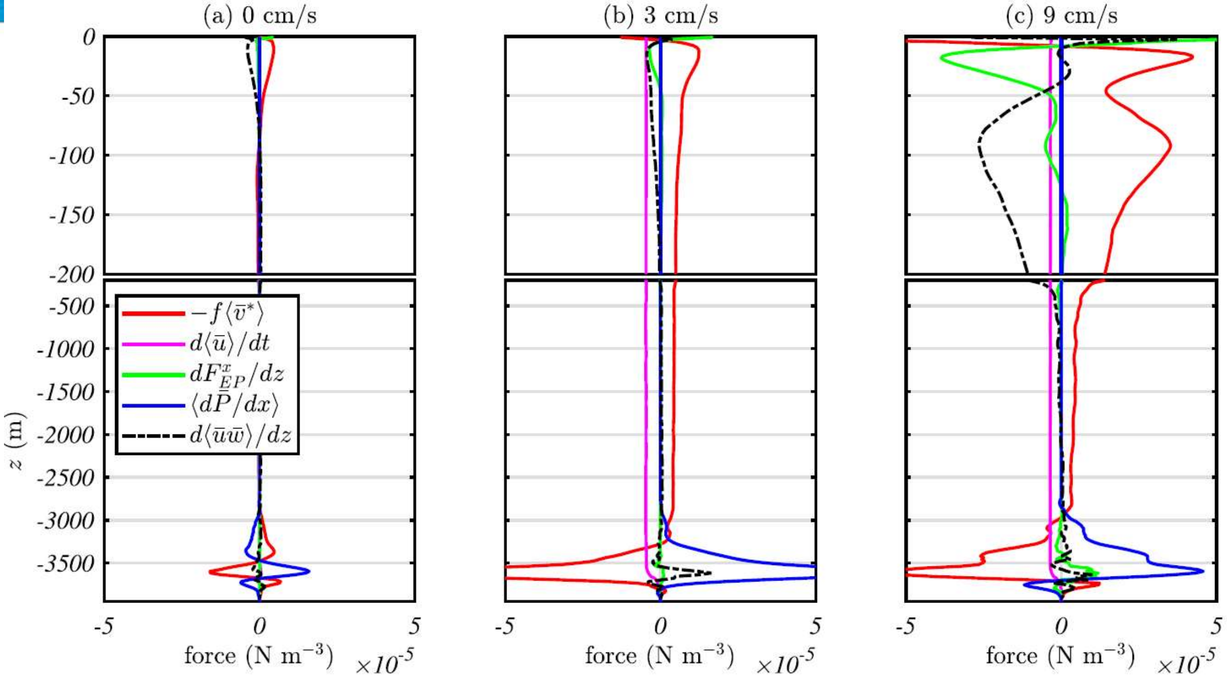
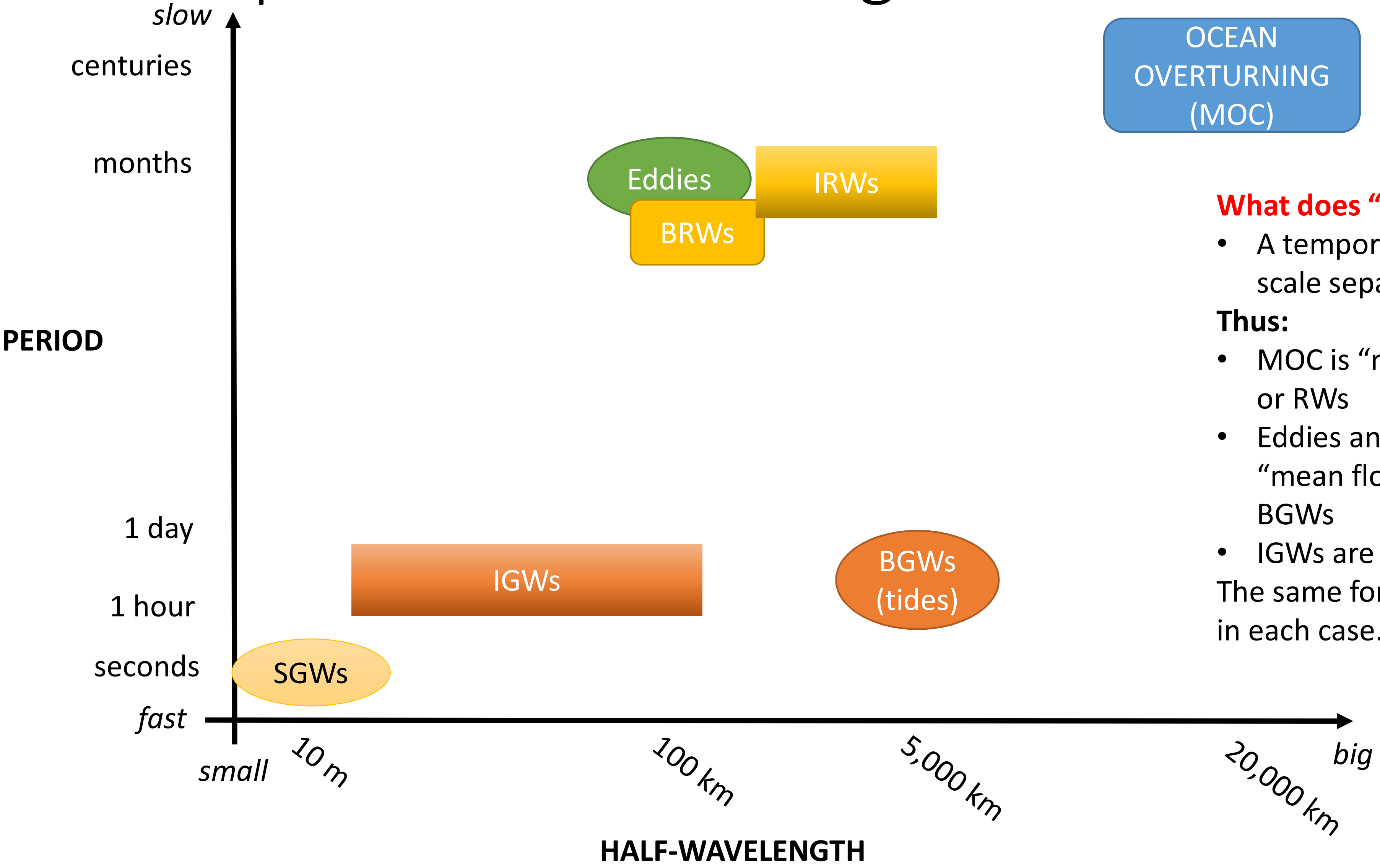


FIG. 10. The domain-averaged zonal momentum balance for the (a) 0, (b) 3, and (c) 9 cm s^{-1} simulations as per (24). The time average is taken over 594 h.

Shakespeare, Callum J., and Andrew McC. Hogg. "On the momentum flux of internal tides." *Journal of Physical Oceanography* 49.4 (2019): 993-1013.

Ocean space-time scale diagram



What does "mean" really mean?

- A temporal and/or spatial scale separation!

Thus:

- MOC is "mean flow" for eddies or RWs
- Eddies and MOC are both "mean flow" for IGWs and BGWs
- IGWs are mean for SGWs

The same formalism can be used in each case.

References

Wave form stress/EP flux:

- Andrews, D. G., and M. E. McIntyre. "Generalized Eliassen-Palm and Charney-Drazin theorems for waves on axismmetric mean flows in compressible atmospheres." *Journal of Atmospheric Sciences* 35.2 (1978): 175-185.
- McIntyre, M. E. "On the ‘wave momentum’ myth." *Journal of Fluid Mechanics* 106 (1981): 331-347.
- Bretherton, Francis P. "Momentum transport by gravity waves." *Quarterly Journal of the Royal Meteorological Society* 95.404 (1969): 213-243.

Applications of EP flux analysis to the ocean

- Shakespeare, Callum J., and Andrew McC. Hogg. "On the momentum flux of internal tides." *Journal of Physical Oceanography* 49.4 (2019): 993-1013.
- Abernathey, R., Marshall, J. & Ferreira, D. Dependence of southern ocean overturning on wind stress. J. Phys. Oceanogr. 41, 22612278 (2011)

Small dissipation approximation for lee waves

- Shakespeare, Callum J., and A. McC Hogg. "On the spurious dissipation of internal waves in ocean circulation models." (2016).